BERNOULLI MEASURES FOR THE TEICHMÜLLER FLOW

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To the memory of Martine Babillot

Abstract. Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. We construct an uncountable family of probability measures on the space $\mathcal{Q}(S)$ of area one holomorphic quadratic differentials over the moduli space for S containing the usual Lebesgue measure. These measures are invariant under the Teichmüller geodesic flow Φ^t, and they are mixing, absolutely continuous with respect to the stable and unstable foliation and exponentially recurrent to a compact set. We show that the critical exponent of the mapping class group equals $6g-6+2m$. Moreover, this critical exponent coincides with the logarithmic asymptotic for the number of closed Teichmüller geodesics in moduli space which meet a sufficiently large compact set.

1. Introduction

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g-3+m \geq 2$, i.e. that S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface S *nonexceptional*. Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The smooth fibre bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all *holomorphic quadratic differentials* of area one can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric*. The *Teichmüller geodesic flow* Φ^t on $\mathcal{Q}^1(S)$ commutes with the action of the *mapping class group* $\mathcal{M}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S. Thus this flow descends to a flow on the quotient $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\mathcal{M}(S)$, again denoted by Φ^t.

For a quadratic differential $q \in \mathcal{Q}^1(S)$ define the *unstable manifold* $W^u(q) \subset \mathcal{Q}^1(S)$ to be the set of all quadratic differentials whose vertical measured geodesic lamination is a multiple of the vertical measured geodesic lamination for q. Then $W^u(q)$ is a submanifold of $\mathcal{Q}^1(S)$ which projects homeomorphically onto $\mathcal{T}(S)$ [HM79]. Similarly, define the *strong stable manifold* $W^{ss}(q)$ to be the set of all quadratic differentials whose horizontal measured geodesic lamination coincides with the horizontal measured geodesic lamination of q. The sets $W^u(q)$ (or $W^{ss}(q)$)

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($q \in \mathcal{Q}^1(S)$) define a foliation of $\mathcal{Q}^1(S)$ which is invariant under the mapping class group and hence projects to a singular foliation on $\mathcal{Q}(S)$ which we call the *unstable foliation* (or the *strong stable foliation*).

There is a Φ^t-invariant probability measure λ on $\mathcal{Q}(S)$ in the Lebesgue measure class which was discovered by Masur [M82] (see also the papers [V82, V86] of Veech which contain a similar construction). This measure is absolutely continuous with respect to the strong stable and the unstable foliation, it is ergodic and mixing, and its metric entropy equals $6g-6+2m$ (note that we use a normalization for the Teichmüller flow which differs from the one in [M82, V86]). Recently Avila, Gouëzel and Yoccoz [AGY05] established that the Lebesgue measure λ_0 on the moduli space of *abelian* differentials is even exponentially mixing, i.e. exponential decay of correlations for Hölder observables holds. The proof of this result uses the fact, independently due to Athreya [A05], that the Teichmüller flow is *exponentially recurrent* to some fixed compact set, i.e. there is a compact set $K \subset \mathcal{Q}(S)$ and a number $c > 0$ such that $\lambda_0\{q \mid \Phi^t q \notin K \text{ for all } t \in [0,T]\} < e^{-cT}/c$ for every $T > 0$.

The main goal of this paper is to construct a subshift of finite type (Ω, T) and a Borel suspension X over (Ω, T) which admits a semi-conjugacy Ξ into the Teichmüller flow. The images under Ξ of the flow invariant measures on X induced by Gibbs equilibrium states for the shift (Ω, T) define an uncountable family of Φ^t-invariant probability measures on $\mathcal{Q}(S)$ including the Lebesgue measure. We summarize their properties as follows.

Theorem 1. *There is an uncountable family of Φ^t-invariant probability measures on $\mathcal{Q}(S)$ including the Lebesgue measure which are mixing, absolutely continuous with respect to the strong stable and the unstable foliation and which moreover are exponentially recurrent to a compact set.*

Denote by d the Teichmüller metric on $\mathcal{T}(S)$. For a fixed point $x \in \mathcal{T}(S)$ and a number $\alpha > 0$, the *Poincaré series* of the mapping class group with exponent α and basepoint x is defined to be the series

$$\sum_{g \in \mathcal{M}(S)} e^{-\alpha d(x, gx)}.$$

The *critical exponent* of $\mathcal{M}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series with exponent α converges [Su79].

For a compact subset K of $\mathcal{Q}(S)$ and for $r > 0$ let $n_K(r)$ be the number of periodic orbits for the Teichmüller geodesic flow on $\mathcal{Q}(S)$ of period at most r which intersect K. We show.

Theorem 2.
(1) *The critical exponent of the mapping class group equals $6g-6+2m$, and the Poincaré series diverges at $6g-6+2m$.*
(2) *For every sufficiently large compact subset K of $\mathcal{Q}(S)$ we have*

$$\lim_{r \to \infty} \frac{1}{r} \log n_K(r) = 6g - 6 + 2m.$$

Theorem 3 is an improvement of a recent result of Bufetov [Bu06], with a different proof. Much earlier, Veech [V86] established a counting result for *all* periodic orbits of the Teichmüller flow. Namely, denote by $n(r)$ the number of all periodic orbits for the Teichmüller flow on $\mathcal{Q}(S)$ of period at most r. Veech showed the existence of a number $k > 6g-6+2m$ such that

$$6g-6+2m \le \lim\inf_{r\to\infty} \frac{1}{r}\log n(r) \le \lim\sup_{r\to\infty} \frac{1}{r}\log n(r) \le k.$$

As a corollary of Theorem C, we deduce that we can take $k = (6g-6+2m)(6g-5+2m)$ which is still far from the conjectured value $6g-6+2m$. On the other hand, in [H06b] we constructed for a surface S of genus g with m punctures and $3g-3+m \ge 4$ and for every compact subset K of $\mathcal{Q}(S)$ a periodic orbit for Φ^t which does not intersect K. Thus if $3g-3+m \ge 4$ then for every compact subset K of $\mathcal{Q}(S)$ we have $n(r) > n_K(r)$ for all sufficiently large r (depending on K). Recently Eskin and Mirzakhani announced that the asymptotic growth of periodic orbits is indeed $6g-6+2m$.

The organization of the paper is as follows. In Section 2 we review some properties of train tracks and geodesic laminations needed in the sequel. In Section 3 we use the *curve graph* of S to show that the restriction of the Teichmüller flow to any invariant compact subset K of $\mathcal{Q}(S)$ is a hyperbolic flow in a topological sense. In Section 4 we use train tracks to construct a special subshift of finite type (Ω, T). In Section 5 we construct a bounded roof function ρ for our subshift of finite type and a semi-conjugacy of the suspension of (Ω, T) with roof function ρ into the Teichmüller flow. This is used in Section 6 to show Theorem 1. In Section 7 we calculate the critical exponent for $\mathcal{M}(S)$, and Section 8 is devoted to the proof of the second part of Theorem 2.

2. Train tracks and geodesic laminations

In this section we summarize some results and constructions from [T79, PH92, H06a] which will be used throughout the paper (compare also [M03]).

Let S be an oriented surface of genus $g \ge 0$ with $m \ge 0$ punctures and where $3g-3+m \ge 2$. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87, O96]. A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons. A geodesic lamination *fills up* S if its complementary regions are all topological discs or once punctured topological discs. A geodesic lamination λ is called *complete* if λ is maximal and can be approximated in the *Hausdorff topology* by simple closed geodesics.

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S with endpoints in the complementary regions of λ which intersects λ nontrivially and transversely. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{ML}$ of all measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to $S^{6g-7+2m} \times (0,\infty)$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations*. There is a continuous symmetric pairing $i : \mathcal{ML} \times \mathcal{ML} \to [0,\infty)$, the so-called *intersection form*, which extends the geometric intersection number between simple closed curves. The measured geodesic lamination $\mu \in \mathcal{ML}$ *fills up* S if its support fills up S. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S.

A *maximal generic train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Every switch is trivalent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. The complementary regions of the train track are trigons, i.e. discs with three cusps at the boundary, or once punctured monogons, i.e. once punctured discs with one cusp at the boundary. We always identify train tracks which are isotopic (see [PH92] for a comprehensive account on train tracks).

A maximal generic train track or a geodesic lamination σ is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is isotopic to the identity and maps σ into τ in such a way that the restriction of the differential of F to the tangent space of σ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to σ a *carrying map* for σ. Write $\sigma \prec \tau$ if the train track or the geodesic lamination σ is carried by the train track τ.

A *transverse measure* on a maximal generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: For every switch s of τ, the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$. The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of an euclidean cone. Via a carrying map, a measured geodesic lamination carried by τ defines a transverse measure on τ, and every transverse measure arises in this way [PH92]. Thus $\mathcal{V}(\tau)$ can naturally be identified with a subset of $\mathcal{ML}$ which is invariant under scaling. A maximal generic train track τ is recurrent if and only if the subset $\mathcal{V}(\tau)$ of $\mathcal{ML}$ has nonempty interior.

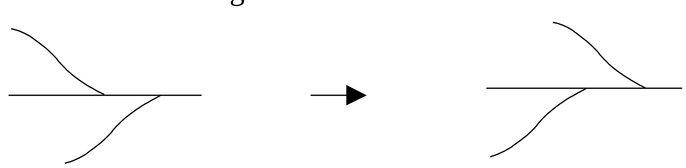
A *tangential measure* μ for a maximal generic train track τ associates to every branch b of τ a nonnegative weight $\mu(b)$ such that for every complementary triangle with sides c_1, c_2, c_3 we have $\mu(c_i) \leq \mu(c_{i+1}) + \mu(c_{i+2})$ (indices are taken modulo

three). The space $\mathcal{V}^*(\tau)$ of all tangential measures on τ has the structure of an euclidean cone. The maximal generic train track τ is called *transversely recurrent* if it admits a tangential measure μ which is positive on every branch [PH92]. There is a one-to-one correspondence between the space of tangential measures on τ and the space of measured geodesic laminations which *hit τ efficiently* (we refer to [PH92] for an explanation of this terminology). With this identification, the pairing $\mathcal{V}(\tau)\times\mathcal{V}^*(\tau)\to[0,\infty)$ defined by $(\mu,\nu)\to\sum_b\mu(b)\nu(b)$ is just the intersection form on $\mathcal{ML}$ [PH92]. A maximal generic train track τ is called *complete* if it is recurrent and transversely recurrent.

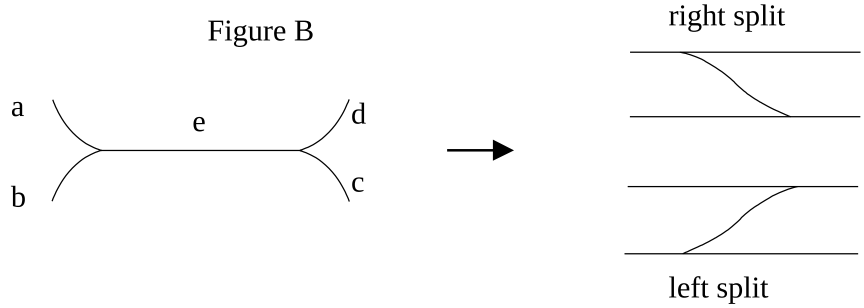
A half-branch $\hat{b}$ in a complete train track τ incident on a switch v of τ is called *large* if every embedded arc of class C^1 containing v in its interior passes through $\hat{b}$. A half-branch which is not large is called *small.* A branch b in a complete train track τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches, and it is large at both of them. A branch is called *small* if each of its two half-branches is small. A branch is called *mixed* if one of its half-branches is large and the other half-branch is small (for all this, see [PH92] p.118).

There are two simple ways to modify a complete train track τ to another complete train track. First, we can *shift* τ along a mixed branch to a train track τ' as shown in Figure A below. If τ is complete then the same is true for τ'. Moreover, a train track or a lamination is carried by τ if and only if it is carried by τ' (see [PH92] p.119). In particular, the shift τ' of τ is carried by τ. Note that there is a natural bijection of the set of branches of τ onto the set of branches of τ'.

Figure A



Second, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure B. Note that a right split at e is uniquely determined by the orientation of S and does not depend on the orientation of e. Using the labels in the figure, in the case of a right split we call the branches a and c *winners* of the split, and the branches b,d are *losers* of the split. If we perform a left split, then the branches b,d are winners of the split, and the branches a,c are losers of the split. The split τ' of a train track τ is carried by τ, and there is a natural choice of a carrying map which maps the switches of τ' to the switches of τ. There is a natural bijection of the set of branches of τ onto the set of branches of τ' which maps the branch e to the diagonal e' of the split. The split of a complete train track is maximal, transversely recurrent and generic, but it may not be recurrent. In the sequel we denote by $\mathcal{TT}$ the set of isotopy classes of complete train tracks on S.



3. The curve graph

In this section we relate Teichmüller geodesics and the Teichmüller flow to the geometry of the *curve graph* $\mathcal{C}(S)$ of S. This curve graph is a metric graph whose vertices are the free homotopy classes of *essential* simple closed curves on S, i.e. curves which are neither contractible nor freely homotopic into a puncture of S. In the sequel we often do not distinguish between an essential simple closed curve and its free homotopy class whenever no confusion is possible. Two such curves are connected in $\mathcal{C}(S)$ by an edge of length one if and only if they can be realized disjointly. Since our surface S is nonexceptional by assumption, the curve graph $\mathcal{C}(S)$ is connected. Any two elements $c, d \in \mathcal{C}(S)$ of distance at least 3 *jointly fill up* S, i.e. they decompose S into topological discs and once punctured topological discs.

By Bers' theorem, there is a number $\chi_0 > 0$ such that for every complete hyperbolic metric on S of finite volume there is a *pants decomposition* of S consisting of $3g - 3 + m$ simple closed geodesics of length at most χ_0. Moreover, for a given number $\chi \geq \chi_0$, the diameter in $\mathcal{C}(S)$ of the (non-empty) set of all simple closed geodesics on S of length at most χ is bounded from above by a universal constant only depending on χ (and on the topological type of S).

As in the introduction, denote by $\mathcal{Q}^1(S)$ the bundle of quadratic differentials of area one over Teichmüller space $\mathcal{T}(S)$. By possibly enlarging our constant $\chi_0 > 0$ as above we may assume that for every $q \in \mathcal{Q}^1(S)$ there is an essential simple closed curve on S whose *q-length*, i.e. the minimal length of a representative of the free homotopy class of α with respect to the singular euclidean metric defined by q, l at most χ_0 (see [R05, R06] and Lemma 2.1 of [H07]). Thus we can define a map $\Upsilon_{\mathcal{Q}} : \mathcal{Q}^1(S) \to \mathcal{C}(S)$ by associating to a quadratic differential q a simple closed curve $\Upsilon_{\mathcal{Q}}(q)$ whose q-length is at most χ_0. By Lemma 2.1 of [H07], if $\Upsilon'_{\mathcal{Q}}$ is any other choice of such a map then we have $d(\Upsilon_{\mathcal{Q}}(q), \Upsilon'_{\mathcal{Q}}(q)) \leq a(\chi_0)$ for all $q \in \mathcal{Q}^1(S)$ where d is the distance function on $\mathcal{C}(S)$. Similarly we obtain a map $\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S)$ by associating to a hyperbolic metric g on S of finite volume a simple closed curve c of g-length at most χ_0. By the discussion in [H06a] there is a constant $L > 1$ such that

$$d(\Upsilon_{\mathcal{T}}(g), \Upsilon_{\mathcal{T}}(h)) \leq L d(g, h) + L \quad \text{for all} \quad g, h \in \mathcal{T}(S). \tag{1}$$

Let $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. By Lemma 2.2 of [H07] (or see the work of Rafi [R05, R06]), we have.

Lemma 3.1. *There is a constant $\chi_1 > 0$ such that $d(\Upsilon_{\mathcal{Q}}(q), \Upsilon_{\mathcal{T}}(Pq)) \le \chi_1$ for all $q \in \mathcal{Q}^1(S)$.*

For a quadratic differential $q \in \mathcal{Q}^1(S)$ the *strong stable manifold* $W^{ss}(q) \subset \mathcal{Q}^1(S)$ is defined as the set of all quadratic differentials of area one whose horizontal measured geodesic lamination coincides *precisely* with the horizontal measured geodesic lamination of q. The *strong unstable* manifold $W^{su}(q)$ is defined as $\mathcal{F}W^{ss}(\mathcal{F}(q))$ where $\mathcal{F}$ is the *flip* $z \to -z$. The map $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ which associates to a quadratic differential its horizontal projective measured geodesic lamination restricts to a homeomorphism of $W^{su}(q)$ onto the open contractible subset of $\mathcal{PML}$ of all projective measured geodesic laminations μ which together with $\pi(-q)$ *jointly fill up* S, i.e. are such that for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $i(\mu, \eta) + i(\pi(-q), \eta) \neq 0$ (note that this makes sense even though the intersection form i is defined on $\mathcal{ML}$ rather than on $\mathcal{PML}$). By definition, the manifolds $W^{ss}(q), W^{su}(q)$ ($q \in \mathcal{Q}^1(S)$) define continuous foliations of $\mathcal{Q}^1(S)$ which are invariant under the action of $\mathcal{M}(S)$ and under the action of the *Teichmüller geodesic flow* Φ^t. Hence they descend to singular foliations on $\mathcal{Q}(S)$ which we denote by the same symbol. Veech [V86] showed that there is a family D of distance functions on strong stable manifolds in $\mathcal{Q}(S)$ such that for a set A of quadratic differentials which has full mass with respect to the natural *Lebesgue measure* on $\mathcal{Q}(S)$ [M82, V86] and every $q \in A$ there is a constant $c(q) > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log D(\Phi^t q, \Phi^t y) \le -c(q) \quad \text{for all} \quad y \in W^{ss}(q). \tag{2}$$

The following proposition is a modified version of this result. Namely, define a closed invariant subset K for a continuous flow Θ^t on a topological space Y to be a *topological hyperbolic set* if there is a distance d on K defining the subspace topology and there is a constant $a > 0$ such that every point $z \in K$ has an open neighborhood U in Y which is homeomorphic to a product $U = U_1 \times U_2$ and such that the following holds.

(1) For all $y \in U_2$, all $x_1, x_2 \in U_1$ such that $(x_i, y) \in K$ $(i = 1, 2)$ and all $t > 0$ we have $d(\Theta^t(x_1, y), \Theta^t(x_2, y)) \le e^{-at}/a$.
(2) For all $x \in U_1$, $y_1, y_2 \in U_2$ with $(x, y_i) \in K$ $(i = 1, 2)$ there is a number $s \in [-1/a, 1/a]$ such that $d(\Theta^{-t}(x, y_1), \Theta^{-t+s}(x, y_2)) \le e^{-at}/a$ for all $t > 0$.

We show.

Proposition 3.2. *Every Φ^t-invariant compact subset K of $\mathcal{Q}(S)$ is a topological hyperbolic set for Φ^t.*

Proof. Let as before $\chi_0 > 0$ be such that for every complete hyperbolic metric on S of finite volume there is a pants decomposition consisting of pants curves of length at most χ_0. Choose a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi_0] \equiv 1$ and $\sigma[2\chi_0, \infty) \equiv 0$. For each $h \in \mathcal{T}(S)$, the number of essential simple closed curves α on S whose hyperbolic length $\ell_h(\alpha)$ (i.e. the length of a geodesic representative of its free homotopy class) does not exceed $2\chi_0$ is bounded from above by a universal

constant not depending on h, and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well. Thus we obtain for every $h \in \mathcal{T}(S)$ a finite Borel measure μ_h on $\mathcal{C}(S)$ by defining $\mu_h = \sum_{c\in\mathcal{C}(S)} \sigma(\ell_h(c))\delta_c$ where δ_c denotes the Dirac mass at c. The total mass of μ_h is bounded from above and below by a universal positive constant, and the diameter of the support of μ_h in $\mathcal{C}(S)$ is uniformly bounded as well. Moreover, the measures μ_h depend continuously on $h \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$ the function $h \to \int f d\mu_h$ is continuous.

The curve graph $(\mathcal{C}(S), d)$ is a hyperbolic geodesic metric space [MM99] and hence it admits a *Gromov boundary* $\partial\mathcal{C}(S)$. For $c \in \mathcal{C}(S)$ there is a complete distance function δ_c on $\partial\mathcal{C}(S)$ of uniformly bounded diameter and there is a number $\kappa > 0$ such that $\delta_c \leq e^{\kappa d(c,a)}\delta_a$ for all $c, a \in \mathcal{C}(S)$. For $h \in \mathcal{T}(S)$ define a distance δ_h on $\partial\mathcal{C}(S)$ by

$$\delta_h(\xi, \zeta) = \int \delta_c(\xi, \zeta) d\mu_h(c)/\mu_h(\mathcal{C}(S)). \tag{3}$$

Clearly δ_h depends continuously on h, moreover the metrics δ_h are equivariant with respect to the action of $\mathcal{M}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$.

Let $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ be the set of all hyperbolic structures whose *systole* is at least ϵ. Let $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection and denote by $K(\epsilon) \subset \mathcal{Q}^1(S)$ the collection of all quadratic differentials whose orbit under the Teichmüller flow Φ^t projects to a Teichmüller geodesic entirely contained in $\mathcal{T}(S)_\epsilon$. By Theorem 2.1 of [H05] and Lemma 3.1, there is a constant $p > 0$ only depending on ϵ such that for every $q \in K(\epsilon)$ the curve $t \to \Upsilon_{\mathcal{Q}}(\Phi^t q)$ is a *p-quasi-geodesic* in $\mathcal{C}(S)$, i.e. we have

$$|t - s|/p - p \leq d(\Upsilon_{\mathcal{Q}}(\Phi^t q), \Upsilon_{\mathcal{Q}}(\Phi^s q)) \leq p|t - s| + p \quad \text{for all} \quad s, t \in \mathbb{R}. \tag{4}$$

For $v \in K(\epsilon)$ define a distance function d^{su} on $W^{su}(v) \cap K(\epsilon)$ as follows. Recall that the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ can be identified with the set of all (unmeasured) minimal geodesic laminations λ on S which fill up S, i.e. which decompose S into topological discs and once punctured topological discs [Kl99, H06a]. If $w \in K(\epsilon)$ then the horizontal measured geodesic lamination of w is *uniquely ergodic*, i.e. it admits a unique transverse measure up to scale, and its support $\nu(w)$ is minimal and fills up S [M82]. Moreover, the p-quasi-geodesic $t \to \Upsilon_{\mathcal{Q}}(\Phi^t w)$ converges in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ to $\nu(w)$. In particular, for every $v \in \mathcal{Q}^1(S)$ there is a natural injection $\iota_v : W^{su}(v) \cap K(\epsilon) \to \partial\mathcal{C}(S)$, and this map is continuous (compare the discussion of the topology on $\partial\mathcal{C}(S)$ in [H06a]). Its image is a locally compact subset of $\partial\mathcal{C}(S)$.

For $y, z \in W^{su}(v) \cap K(\epsilon)$, a *chain* from y to z is a finite collection $y_0 = y, y_1, \ldots, y_k = z \subset W^{su}(v) \cap K(\epsilon)$. For such a chain c, write

$$\rho(c) = \frac{1}{2}\sum_{i=1}^{k}(\delta_{Py_{i-1}}(\iota_v(y_{i-1}), \iota_v(y_i)) + \delta_{Py_i}(\iota_v(y_{i-1}), \iota_v(y_i))) \tag{5}$$

and let $d^{su}(y, z) = \inf\{\rho(c) \mid c \text{ is a chain from } y \text{ to } z\}$. Note that by the properties of the distance functions δ_h $(h \in \mathcal{T}(S))$ the function d^{su} is a distance function on $W^{su}(v) \cap K(\epsilon)$ which does not depend on v. Locally near v, the map ι_v is

uniformly bilipschitz with respect to d^{su} and the distance function δ_{Pv} on $\partial\mathcal{C}(S)$. The resulting family d^{su} of metrics on the intersection with $K(\epsilon)$ of strong unstable manifolds is invariant under the action of the mapping class group. Moreover, it is continuous in the following sense. If $v_i \in K(\epsilon), w_i \in W^{su}(v_i) \cap K(\epsilon)$ and if $v_i \to v, w_i \to w$ then $d^{su}(v_i, w_i) \to d^{su}(v, w)$. Similary we obtain a continuous $\mathcal{M}(S)$-invariant family d^{ss} of distance functions on the intersections with $K(\epsilon)$ of the strong stable manifolds.

Since for every $y \in K(\epsilon)$ the assignment $t \to \Upsilon_{\mathcal{Q}}(\Phi^t y)$ is a p-quasi-geodesic in $\mathcal{C}(S)$ where $p > 0$ only depends on ϵ, we obtain from hyperbolicity of $\mathcal{C}(S)$, the definition of the metrics δ_c, Lemma 3.2 and inequality (1) above the existence of numbers $a > 0, b > 0$ only depending on ϵ such that the flow Φ^t satisfies the following contraction property.

(1) $d^{ss}(\Phi^t v, \Phi^t w) \leq be^{-at} d^{ss}(v, w)$ for all $t \geq 0, w \in W^{ss}(v) \cap K(\epsilon)$.
(2) $d^{su}(\Phi^{-t} v, \Phi^{-t} w) \leq be^{-at} d^{su}(v, w)$ for all $t \geq 0, w \in W^{su}(v) \cap K(\epsilon)$.

Now the assignment which associates to a quadratic differential q its horizontal and its vertical measured foliation is continuous and therefore for every $q \in K(\epsilon)$ there is an open neighborhood U_q of q in $\mathcal{Q}^1(S)$, an open neighborhood V_1 of q in $W^{ss}(q)$, an open neighborhood V_2 of q in $W^{su}(q)$, a number $c > 0$ and a homeomorphism $\psi : U_q \to V_1 \times V_2 \times (-c, c)$ which associates to a quadratic differential $z \in U_q$ the triple $\psi(z) = (\psi_1(z), \psi_2(z), s(z))$. Here $\psi_1(z)$ equals the quadratic differential in $V_1 \cap W^u(z)$ and $s(z) \in \mathbb{R}$ is such that $\psi_1(z) \in \Phi^{s(z)} W^{su}(z)$, and $\psi_2(z)$ equals the quadratic differential in $V_2 \cap W^s(z)$. The homeomorphism ψ then defines a box metric ρ_q on $U_q \cap K(\epsilon)$ by

$$\rho_q(u, z) = d^{ss}(\psi_1(u), \psi_1(z)) + d^{su}(\psi_2(u), \psi_2(z)) + |s(z) - s(u)|. \tag{6}$$

Let $\Pi : \mathcal{Q}^1(S) \to \mathcal{Q}(S)$ be the canonical projection; the map Π is open. The projection $K_0(\epsilon) = \Pi K(\epsilon)$ of $K(\epsilon)$ to $\mathcal{Q}(S)$ is compact. Choose a finite covering of $K_0(\epsilon)$ by open sets $U_1, \dots, U_\ell$ which are images under Π of open subset $U_{q_1}, \dots, U_{q_\ell}$ of the above form ($q_i \in K(\epsilon)$). For every i let $\rho(q_i)$ be the box metric on U_{q_i} and define a distance ρ_i on $\Pi(U_{q_i})$ by $\rho_i(z_1, z_2) = \inf\{\rho(q_i)(\tilde{z}_1, \tilde{z}_2) \mid \tilde{z}_j \in U_{q_i}, \Pi(\tilde{z}_j) = z_j\, (j = 1, 2)\}$. Let $\{f_i\}$ be a partition of unity for this covering and define a distance ρ on $K_0(\epsilon)$ by $\rho = \sum_i f_i \rho_i$. The distance function ρ has the properties required in the definition of a topological hyperbolic set. The proposition now follows from the fact that $K_0(\epsilon) \subset K_0(\delta)$ for $\epsilon > \delta$ and that $\cup_{\epsilon > 0} K_0(\epsilon)$ contains every compact Φ^t-invariant subset of $\mathcal{Q}(S)$. $\square$

4. A symbolic system

In this section we use train tracks to construct a subshift of finite type which we use in the following sections to study the Teichmüller geodesic flow.

Define a *numbered complete train track* to be a complete train track τ together with a numbering of the branches of τ. Since a mapping class which preserves a train track $\tau \in \mathcal{TT}$ as well as each of its branches is the identity (compare the proof

of Lemma 3.3 of [H06a]), the mapping class group $\mathcal{M}(S)$ acts *freely* on the set $\mathcal{NT}$ of all isotopy classes of numbered complete train tracks on S.

If a complete train track τ' is obtained from τ by a single shift, then a numbering of the branches of τ induces a numbering of the branches of τ' in such a way that the branch with number i is large (or mixed or small) in τ if and only if this is the case for the branch with number i in τ'. Define the *numbered class* of a numbered train track τ to consist of all numbered train tracks which can be obtained from τ by a sequence of numbered shifts. The mapping class group naturally acts on the set of all numbered classes. We have.

Lemma 4.1. *The action of $\mathcal{M}(S)$ on the set of numbered classes is free.*

Proof. Let $\tau \in \mathcal{NT}$ be a numbered complete train track and let $\varphi \in \mathcal{M}(S)$ be such that $\varphi(\tau)$ is contained in the numbered class of τ. We have to show that φ is the identity. For this note that since $\varphi(\tau)$ can be obtained from τ by a sequence of numbered shifts, there is a natural bijection between the complementary components of τ and the complementary components of $\varphi(\tau)$. These complementary components are trigons, i.e. topological discs with three sides of class C^1 which meet at the cusps of the component, and once punctured monogons, i.e. once punctured discs with a single side whose endpoints meet at the cusp of the component. Thus if φ is not the identity then with respect to the natural identification of complementary components, φ induces a nontrivial permutation of the complementary components of τ.

Now for each fixed complementary component C of τ, the train track τ can be modified with a sequence of shifts to a train track τ' with the property that the boundary of the complementary component C' of τ' corresponding to C consists entirely of small and large branches. But this just means that the complementary component C is determined by the small and large branches in its boundary. Since $\varphi(\tau)$ is contained in the numbered class of τ, the map φ preserves the large and the small branches of τ and hence it maps C to itself. This implies that φ is the identity and shows the lemma. $\square$

Define a *numbered combinatorial type* to be an orbit of a numbered class under the action of the mapping class group. Thus by definition, the set of numbered combinatorial types equals the quotient of the set of numbered classes under the action of the mapping class group. Let $\mathcal{E}_0$ be the set of all numbered combinatorial types. If the numbered combinatorial type defined by a numbered train track τ is contained in a subset $\mathcal{E}$ of $\mathcal{E}_0$, then we say that τ is *contained* in $\mathcal{E}$ and we write $\tau \in \mathcal{E}$.

If the complete train track τ' can be obtained from a complete train track τ by a single split, then a numbering of the branches of τ naturally induces a numbering of the branches of τ' and therefore such a numbering defines a *numbered split*. In particular, we can define a *split* of a numbered class x to be a numbered class x' with the property that there are representatives τ, τ' of x, x' such that τ' can be obtained from τ by a single numbered split in this sense. If τ' is obtained from τ by a single split at a large branch with number e then a large branch $b \neq e$ in τ

is large in τ' as well. Thus we can define a *full split* of a numbered class x to be a numbered class x' with the property that there are representatives τ, τ' of x, x' such that τ' can be obtained from τ by splitting τ at each large branch once. A *numbered splitting and shifting sequence* is a sequence (τ_i) of numbered complete train tracks such that for each i, the numbered train track τ_{i+1} can be obtained from a shift of τ_i by a single numbered split. Similarly, a *full numbered splitting and shifting sequence* is a sequence (τ_i) of numbered complete train tracks such that for each i, the numbered train track τ_{i+1} can be obtained from a shift of τ_i by a single full numbered split.

We say that a numbered combinatorial type $c \in \mathcal{E}_0$ is *splittable* to a numbered combinatorial type c' if there is a numbered train track $\tau \in c$ which can be connected to a numbered train track $\tau' \in c'$ by a full numbered splitting and shifting sequence.

Recall from [PH92, H06a, H06b] the definition of a train track in *standard form* for some *framing* (or *marking* in the terminology of Masur and Minsky [MM99]) of our surface S. Each such train track is complete. Define a *twist connector* in a train track to be an embedded closed trainpath of length 2 consisting of a large branch and a small branch. We call a train track τ to be in *special standard form* if it satisfies the following properties.

(1) Every large branch of τ is contained in a twist connector.
(2) The union of all twist connectors in τ is the pants decomposition P of our framing.
(3) τ carries a complete geodesic lamination λ whose minimal components are the pants curves of S and such that every pair of pants of our decomposition contain precisely three leaves of λ spiraling about mutually distinct pairs of boundary curves of the pair of pants.

We have.

Lemma 4.2. *There is a set $\mathcal{E} \subset \mathcal{E}_0$ of numbered combinatorial types with the following properties.*

(1) *For all $x, x' \in \mathcal{E}$, x is splittable to x'.*
(2) *If τ is contained in $\mathcal{E}$ and if (τ_i) is any full numbered splitting and shifting sequence issuing from $\tau_0 = \tau$ then τ_i is contained in $\mathcal{E}$ for all $i \geq 0$.*
(3) *For every train track σ in special standard form for some framing of S there is a numbering of the branches of σ such that the resulting numbered train track is contained in $\mathcal{E}$.*

Proof. Let $\tau \in \mathcal{NT}$ be any numbered train track in special standard form for some framing of S and let $\mathcal{E} \subset \mathcal{E}_0$ be the set of numbered combinatorial types of all complete train tracks which can be obtained from τ by a full numbered splitting and shifting sequence. By construction, the set $\mathcal{E}$ has property (2) in the lemma.

Let η be any numbered train track which contains a twist connector c consisting of a large branch e and a small branch b. Then the train track η_1 obtained from η by a split at e with b as a winner is contained in the numbered class which is obtained from η by exchanging the numbers of e and b. This means the following. Let

$e_1, \dots, e_k$ be the large branches of η different from e and assume that the numbered train track σ is obtained from η by one split at each large branch $e_1, \dots, e_k$. Then there is a numbered combinatorial type obtained from the numbered combinatorial type of τ by a full numbered split and which is obtained from the type of σ by a permutation of the numbering.

Let $y \in \mathcal{E}$ and let $\sigma \in \mathcal{NT}$ be a numbered train track representing y. It follows from the considerations in [H06b] that there is a numbered splitting sequence connecting σ to a numbered train track σ' which can be obtained from a point in the $\mathcal{M}(S)$-orbit of τ by a sequence of shifts and by a permutation of the numbering. Our above consideration shows that σ can also be connected with a full numbered splitting and shifting sequence to a numbered train track which can be obtained from a point in the $\mathcal{M}(S)$-orbit of τ by a permutation of the numbering. Since the collection of all permutations of the numbering of τ obtained in this way clearly forms a group, the train track σ can be connected to a train track of the same numbered combinatorial type as τ by a full numbered splitting and shifting sequence. Thus $\mathcal{E}$ has the first property stated in the lemma. The third property follows from [H06b] in exactly the same way. ☐

Let $k > 0$ be the cardinality of a set $\mathcal{E} \subset \mathcal{E}_0$ as in Lemma 4.2 and number the k elements of $\mathcal{E}$ in an arbitrary order. We identify each element of $\mathcal{E}$ with its number. Define $a_{ij} = 1$ if the numbered combinatorial type i can be split with a single full numbered split to the numbered combinatorial type j and define $a_{ij} = 0$ otherwise. The matrix $A = (a_{ij})$ defines a *subshift of finite type*. The phase space of this shift is the set of biinfinite sequences $\Omega \subset \prod_{i=-\infty}^{\infty}\{1, \dots, k\}$ with the property that $(x_i) \in \Omega$ if and only if $a_{x_i x_{i+1}} = 1$ for all i. Every biinfinite full numbered splitting and shifting sequence $(\tau_i) \subset \mathcal{NT}$ contained in $\mathcal{E}$ defines a point in Ω. Vice versa, since by Lemma 4.1 the action of $\mathcal{M}(S)$ on the set of numbered classes of train tracks is free, a point in Ω determines an $\mathcal{M}(S)$-orbit of biinfinite full numbered splitting and shifting sequences which is unique up to replacing the train tracks in the sequence by shift equivalent train tracks. We say that such a numbered splitting and shifting sequence *realizes* (x_i).

The shift map $T : \Omega \to \Omega, T(x_i) = (x_{i+1})$ acts on Ω. For $m > 0$ write $A^m = (a_{ij}^{(m)})$; the shift T is *topologically transitive* if for all i, j there is some $m > 0$ such that $a_{ij}^{(m)} > 0$. Namely, if we define a finite sequence $(x_i)_{0 \le i \le m}$ of points $x_i \in \{1, \dots, k\}$ to be *admissible* if $a_{x_i x_{i+1}} = 1$ for all i then $a_{ij}^{(m)}$ equals the number of all admissible sequences of length m connecting i to j [Mn87]. The following observation is immediate from the definitions.

Lemma 4.3. *The shift (Ω, T) is topologically transitive.*

Proof. Let $i, j \in \{1, \dots, k\}$ be arbitrary. By Lemma 4.2, there is a nontrivial finite full numbered splitting and shifting sequence $\{\tau_i\}_{0 \le i \le m} \subset \mathcal{NT}$ connecting a train track τ_0 of numbered combinatorial type $i \in \mathcal{E}$ to a train track τ_m of numbered combinatorial type j. This splitting and shifting sequence then defines an admissible sequence $(x_i)_{0 \le i \le m} \subset \mathcal{E}$ connecting i to j. ☐

The subshift of finite type defined by the matrix $A = (a_{ij})$ is *topologically mixing* if there is some $m > 0$ such that $a_{ij}^{(m)} > 0$ for all $i, j \in \{1, \dots, k\}$. The next simple lemma will imply that the shift defined by numbered splitting and shifting sequences as above is topologically mixing.

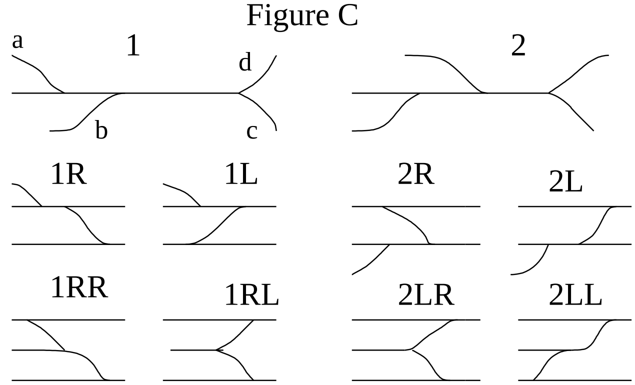
Lemma 4.4. *Let (Ω, T) be a topologically transitive subshift of finite type for the finite alphabet $\{1, \dots, k\}$ and with transition matrix $A = (a_{ij})$. If there are $i, j \le k$ such that $a_{ij} = 1 = a_{ij}^{(2)}$ then (Ω, T) is topologically mixing.*

Proof. Let (Ω, T) be a topologically transitive subshift of finite type as in the lemma with (k, k)-transition matrix $A = (a_{ij})$ and such that for some i, j we have $a_{ij} = 1 = a_{ij}^{(2)}$. Assume to the contrary that (Ω, T) is not topologically mixing. Following [HK95], there is then a number $N \ge 2$ and there is a partition of the set $\{1, \dots, k\}$ into disjoint subsets $P_1, \dots, P_N$ such that for $i \in P_s$ we have $a_{ij} = 1$ only if $j \in P_{s+1}$. Let i, j be as above and let $s \le N$ be such that $i \in P_s$ where indices are taken modulo N. From our assumption $a_{ij} = 1$ we conclude that $j \in P_{s+1}$, on the other hand it follows from $a_{ij}^{(2)} = 1$ that also $j \in P_{s+2}$. In other words, $s + 1 = s + 2$ mod N and therefore $N = 1$ which contradicts our assumption that the shift is not topologically mixing. This shows the lemma. $\square$

Corollary 4.5. *The subshift of finite type defined by the set $\mathcal{E}$ of numbered combinatorial types of complete numbered train tracks on S is topologically mixing.*

Proof. By Lemma 4.4, it is enough to find some numbered combinatorial types $i, j \in \{1, \dots, k\}$ such that our transition matrix $A = (a_{ij})$ satisfies $a_{ij} = 1 = a_{ij}^{(2)}$. However, such numbered combinatorial types of train tracks are shown in Figure C below. Namely, the train tracks $1RR$ and $2R$ are numbered shift equivalent, and the same is true for the train tracks $1L$ and $2LL$. To complete the proof of the corollary, simply observe that any large branch in a complete train track τ incident on a switch v with the property that at least one of the branches of τ which is incident on v and small at v is *not* a small branch defines a configuration as in Figure C. In particular, every large branch of a train track in special standard form for some framing of S is contained in such a configuration (see [PH92], H06a). $\square$

Figure C



5. Symbolic dynamics for the Teichmüller flow

In this section we relate the subshift of finite type (Ω, T) constructed in Section 4 to the extended Teichmüller flow. As in Section 2, for $\tau \in \mathcal{NT}$ denote by $\mathcal{V}(\tau)$ the convex cone of all transverse measures on τ. Recall that $\mathcal{V}(\tau)$ coincides with the space of all measured geodesic laminations which are carried by τ. In particular, if τ' is shift equivalent to τ then we have $\mathcal{V}(\tau) = \mathcal{V}(\tau')$. The projectivization of $\mathcal{V}(\tau)$ can naturally be identified with the space $\mathcal{PM}(\tau)$ of all projective measured geodesic laminations which are carried by τ. Then $\mathcal{PM}(\tau)$ is a *compact* subset of the compact space $\mathcal{PML}$ of all projective measured geodesic laminations on S. If $(\tau_i)_{0\leq i} \subset \mathcal{NT}$ is any numbered splitting and shifting sequence then $\emptyset \neq \mathcal{PM}(\tau_{i+1}) \subset \mathcal{PM}(\tau_i)$ and hence $\cap_i \mathcal{PM}(\tau_i)$ is well defined and non-empty. Thus by the considerations in Section 4, every sequence $(x_i) \in \Omega$ determines an orbit of the action of $\mathcal{M}(S)$ on the space of compact subsets of $\mathcal{PML}$. We call the sequence $(x_i) \in \Omega$ *uniquely ergodic* if $\cap_i \mathcal{PM}(\tau_i)$ consists of a single *uniquely ergodic* point, whose support λ fills up S and admits a unique transverse measure up to scale.

Let $\mathcal{U} \subset \Omega$ be the set of all uniquely ergodic sequences. We define a function $\rho : \mathcal{U} \to \mathbb{R}$ as follows. For $(x_i) \in \mathcal{U}$ choose a full numbered splitting and shifting sequence $(\tau_i) \subset \mathcal{NT}$ which realizes (x_i). By the definition of a uniquely ergodic sequence there is a distinguished uniquely ergodic measured geodesic lamination μ which is carried by each of the train tracks τ_i and such that the maximal weight disposed by this measured geodesic lamination on a *large* branch of τ_0 equals one. Note that if η is shift equivalent to τ_0 then the maximal weight that μ disposes on a large branch of η equals one as well, i.e. this normalization only depends on the numbered class of τ_0. Define $\rho(x_i) \in \mathbb{R}$ by the requirement that the maximal mass that the measured geodesic lamination $e^{\rho(x_i)}\mu$ disposes on a large branch of τ_1 equals one. By equivariance under the action of the mapping class group, the number $\rho(x_i)$ only depends on the sequence (x_i). In other words, ρ is a function defined on $\mathcal{U}$. We have.

Lemma 5.1. *The function $\rho : \mathcal{U} \to \mathbb{R}$ is continuous.*

Proof. Let $(x_i) \in \mathcal{U}$ and let $\epsilon > 0$. By the definition of the topology on our shift space it suffices to show that there is some $j \geq 0$ such that

$$|\rho(y_i) - \rho(x_i)| \leq 2\epsilon$$

whenever $(y_i) \in \mathcal{U}$ is such that $x_i = y_i$ for $0 \leq i \leq j$. For this let (τ_i) be a numbered full splitting and shifting sequence which realizes (x_i). Then (τ_i) defines a measured geodesic lamination λ which is carried by τ_1 and such that the maximal weight which is disposed by λ on any large branch of τ_1 equals one. By definition, $\rho(x_i)$ equals the logarithm of the maximal weight which is disposed by λ on a large branch of τ_0.

Now the space $\mathcal{V}(\tau_1)$ of all transverse measures for τ_1 equipped with the topology as a family of weights on the branches of τ_1 coincides with a closed subset of the set of all measured geodesic laminations equipped with the weak* topology. Thus there is a neighborhood U of λ in the space $\mathcal{ML}$ of all measured geodesic laminations such that for every $\nu \in U \cap \mathcal{V}(\tau_1)$ the logarithm of the maximal weight of a large

branch of τ_1 defined by ν is contained in the interval $(-\epsilon, \epsilon)$ and that the logarithm of the maximal weight that ν disposes on a large branch of τ_0 is contained in the interval $(\rho(x_i) - \epsilon, \rho(x_i) + \epsilon)$. On the other hand, for every $j > 0$ the set $\mathcal{V}(\tau_j)$ of all measured geodesic laminations which are carried by τ_j is a closed cone in $\mathcal{ML}$ containing λ and we have $\mathcal{V}(\tau_j) \subset \mathcal{V}(\tau_1)$ for $j \geq 1$ and $\cap_j \mathcal{V}(\tau_j) = \mathbb{R}\lambda$. As a consequence, there is some $j_0 > 0$ such that $\mathcal{V}(\tau_j) \subset U$. This implies that the value of ρ on the intersection with $\mathcal{U}$ of the cylinder $\{(y_j) \mid y_i = x_i \text{ for } i \leq j_0\}$ is contained in $(\rho(x_i) - 2\epsilon, \rho(x_i) + 2\epsilon)$ which shows the lemma. □

The next lemma gives additional information on the function ρ.

Lemma 5.2. (1) *The function $\rho : \mathcal{U} \to \mathbb{R}$ is nonnegative and uniformly bounded from above.*

(2) *There is a number $p > 0$ such that for every $(x_i) \in \mathcal{U}$ we have*

$$\sum_{j=0}^{p-1} \rho(T^j(x_i)) > 0.$$

(3) *If the support of a measured geodesic lamination defined by $(x_i) \in \mathcal{U}$ is maximal then $\rho(x_i) > 0$.*

Proof. We have to show that there is a number $s > 0$ such that $0 \leq \rho(x_i) \leq s$ for every $(x_i) \in \mathcal{U}$. To show the first inequality, choose a numbered full splitting and shifting sequence $(\tau_i) \subset \mathcal{NT}$ which realizes (x_i). We assume that τ_1 is obtained from τ_0 by a single split at each large branch of τ_0. Using the above notations, let λ^+ be the uniquely ergodic measured geodesic lamination defined by (τ_i). Then λ^+ is carried by each of the train tracks τ_i, and the maximal weight of a large branch of τ_0 for the transverse measure μ on τ_0 defined by λ^+ equals one.

If μ_1 is the transverse measure on τ_1 defined by the measured geodesic lamination λ^+ then for every large branch e of τ_0 we have $\mu(e) = \mu_1(e') + \mu_1(b) + \mu_1(d)$ where b, d are the neighbors of the diagonal e' of our split in τ_1 which are the images of the losing branches of the split under the natural identification of the branches of τ_0 with the branches of τ_1. Moreover, we have $\mu(a) = \mu_1(a)$ for every branch $a \neq e$ of τ_0 and the corresponding branch of τ_1, and the measure μ_1 is nonnegative. In particular, the μ_1-weight of any branch of τ_1 does not exceed the μ-weight of the corresponding branch of τ_0. Since for every transverse measure ν on a complete train track τ the maximum of the ν-weights of the branches of τ is assumed on a large branch [PH92] we conclude that our function ρ is non-negative.

Let $e_1, \dots, e_k$ be the large branches in τ_0 with $\mu(e_i) = 1$ and let a_i, c_i be the branches of τ_0 which are incident on the endpoints of e_i and which are winners of the split. Let moreover b_i, d_i be the losing branches of the split chosen in such a way that the branch b_i is incident on the same endpoint of e_i as a_i. Denote by a_i', b_i' the branches in τ_1 corresponding to a_i, c_i. If there is a large branch in τ_1 of weight one with respect to the measure μ_1 then necessarily this branch is among the branches a_i', c_i', so assume that a_1' is such a branch. Then the μ-weight of the branch b_1 vanishes and hence $\rho(x_i) > 0$ if the support of the measured geodesic lamination λ^+ is maximal. This shows the third part of our lemma.

If λ^+ is any uniquely ergodic measured geodesic lamination which fills up S then the subtrack σ_0 of τ_0 of all branches of positive μ-weight is *large*, i.e. its complementary components are all topological discs or once punctured topological discs. A branch b_1 of τ_0 with $\mu(b_1) = 0$ is contained in the interior of a complementary component of σ_0 which contains both branches a_1, e_1 in its boundary. By the considerations in [H06c], the number of consecutive splits of τ_0 of this form at a large branch contained in the side of a complementary region of σ is bounded from above by the number of branches contained in this side. This just means that there is a universal number $p > 0$ such that $\sum_{j=0}^{p-1} \rho(T^j(x_i)) > 0$ and complete the proof of the second part of our lemma.

On the other hand, by the switch condition for transverse measures on a train track τ, the maximum of the μ_1-weights of the two winning branches a, c of a split of τ_0 at a large branch e is not smaller than half the μ-weight of e. Since the maximum of the weight of a transverse measure on a complete train track τ is assumed on a large branch, our function ρ is uniformly bounded. This completes the proof of the lemma. □

Following [H06c] we call a finite admissible sequence $(x_i)_{0\leq i\leq m} \subset \mathcal{E}$ *tight* if for one (and hence every) numbered splitting and shifting sequence $\{\tau_i\}_{0\leq i\leq m}$ realizing (x_i) the natural carrying map $\tau_m \to \tau_0$ maps every branch b of τ_m *onto* τ_0. By the definition of $\mathcal{E}$, by Lemma 4.2 and by the considerations in Section 5 of [H06c], tight finite admissible sequences exist. Call a biinfinite sequence $(x_j) \in \Omega$ *normal* if every finite admissible sequence occurs in (x_j) infinitely often in forward and backward direction. The next lemma follows from the considerations in [K85]. For this we define a measured geodesic lamination λ on S to be *recurrent* if for one (and hence every) Teichmüller geodesic γ defined by a quadratic differential with horizontal lamination λ there is a compact subset K of moduli space with the property that the intersection of the projection of γ to moduli space with the compact K is unbounded. By the results of Masur [M82], a recurrent lamination is uniquely ergodic and its support fills up S. We have.

Lemma 5.3. *Let $(x_i) \in \Omega$ be normal and let $\{\tau_i\} \subset \mathcal{NT}$ be a numbered splitting and shifting sequence which realizes (x_i); then $\cap_i \mathcal{PM}(\tau_i)$ consists of a single recurrent projective measured geodesic lamination.*

Proof. Let $(x_i) \in \Omega$ be normal and let $\{\tau_i\}_{0\leq i}$ be a numbered splitting and shifting sequence defined by (x_i) and the choice of a numbered train track τ_0 of numbered combinatorial type x_0. Then for every i there is a natural projective linear map $A_i : \mathcal{PM}(\tau_{i+1}) \to \mathcal{PM}(\tau_i)$. Following [K85], this map can be described as follows.

Let n be the number of branches of a complete train track on S. If the train track τ_{i+1} is obtained from τ_i by a single split at a large branch e then every measured geodesic lamination λ which is carried by τ_{i+1} defines both a transverse measure μ_{i+1} on τ_{i+1} and a transverse measure μ_i on τ_i. If a, c are the losers of the split connecting τ_i to τ_{i+1} and if e' is the diagonal branch of the split, then we have $\mu_i(e) = \mu_{i+1}(e') + \mu_{i+1}(a) + \mu_{i+1}(c)$, i.e. the convex cone $\mathcal{V}(\tau_{i+1})$ of transverse measures on τ_{i+1} is mapped to the convex cone $\mathcal{V}(\tau_i)$ of transverse measures on τ_i by the product C_i of two elementary (n, n)-matrices.

For a column vector $x \in \mathbb{R}^n$ write $|x| = \sum_s |x_s|$. Viewing a measure in the cone $V(\tau_i)$ as a point in $\mathbb{R}^n$ we denote by $W(\tau_i)$ the intersection of $V(\tau_i)$ with the sphere $\{x \in \mathbb{R}^n \mid |x| = 1\}$. The set $W(\tau_i)$ can be identified with the projectivization $\mathcal{PM}(\tau_i)$ of $V(\tau_i)$ and is homeomorphic to a closed ball in $\mathbb{R}^{6g-7+2m}$. For all $i > 0$ the projective linear map $A_i : \mathcal{PML}(\tau_{i+1}) \to \mathcal{PML}(\tau_i)$ can naturally be identified with the map $x \in W(\tau_{i+1}) \to C_i(x)/|C_i(x)| \in W(\tau_i)$ (see [K85]). The Jacobian of this map is bounded from above by one at every point of $W(\tau_{i+1})$ [K85].

Following the reasoning in [K85], it is enough to show that there is a number $k \geq 1$ and there are infinitely many j such that for each of the transformations $V(\tau_{j+k}) \to V(\tau_j)$ $(j > 0)$ which is given by products $C_{j,k}$ of $2k$ elementary (n,n)-matrices, every column vector is added to every other column vector at least once. Namely, in this case the Jacobians of the projectivizations of the maps $C_{i,k}$ are bounded from above by a number $\alpha < 1$ which is independent of j. Then the Jacobians of the projectivized transformations $W(\tau_i) \to W(\tau_0)$ tend to zero as $i \to \infty$ uniformly on $W(\tau_i)$. Moreover, the maps $W(\tau_i) \to W(\tau_0)$ are uniformly quasiconformal and therefore their pointwise dilatations tend to zero as well (see [K85]). In particular, the intersection $\cap_i \mathcal{PM}(\tau_i)$ consists of a single point λ. Moreover, for every i the point λ is contained in the *interior* of the convex polyhedron $\mathcal{PM}(\tau_i)$ and hence the transverse measure defined by λ on τ_i is positive on every branch of τ_i (compare the beautiful argument in [K85]).

Now for every tight admissible sequence $(y_\ell)_{0\leq \ell\leq k}$ and for every numbered splitting and shifting sequence $\{\tau_i\}_{j\leq i\leq k+j}$ which realizes $(y_\ell)_{0\leq\ell\leq k}$, the transformation $\mathcal{V}(\tau_{j+k}) \to \mathcal{V}(\tau_j)$ has the above property. As a consequence, if $(x_i) \in \Omega$ is normal and if $\{\tau_i\}$ is a splitting and shifting sequence which realizes (y_i) then $\cap_i \mathcal{PM}(\tau_i)$ consists of a single point λ.

Since $\cap_i \mathcal{PM}(\tau_i)$ contains with every $\mu \in \mathcal{PML}$ every projective measured geodesic lamination with the same support, the projective measured geodesic lamination λ is necessarily uniquely ergodic. We are left with showing that it also fills up S. For this call a geodesic lamination λ *complete* if λ is maximal and if moreover λ can be approximated in the Hausdorff topology by simple closed geodesics. The space $\mathcal{CL}(\tau_i)$ of *complete* geodesic laminations carried by τ_i is a compact non-empty subset of the compact space $\mathcal{CL}$ of all complete geodesic laminations on S (compare [H06a]) and consequently $K = \cap_i \mathcal{CL}(\tau_i)$ is non-empty. Every projective measured geodesic lamination whose support is contained in a lamination from the set K is contained in $\cap_i \mathcal{PM}(\tau_i)$. Since the intersection $\cap_i \mathcal{PM}(\tau_i)$ consists of the unique point λ, the laminations from the set K contain a single minimal component ζ which is the support of λ. If ζ does not fill up S then there is a simple closed curve c on S which intersects a complete geodesic lamination $\nu \in K$ in a finite number of points. By the considerations in [H06a] we can find a sequence $\{c_i\}$ of simple closed curves on S which approximate ν in the Hausdorff topology such that the intersection numbers $i(c_i, c)$ are bounded from above by a universal constant $a > 0$. Moreover, for every $j > 0$ there is some $i(j) > 0$ such that the train track τ_j carries the curve $c_{i(j)}$. As a consequence, there is a splitting and shifting sequence connecting τ_j to a train track $\eta_{i(j)}$ which contains $c_{i(j)}$ as a *vertex cycle* [H06b]. This means that the transverse measure on $\eta_{i(j)}$ defined by a carrying map $c_{i(j)} \to \eta_{i(j)}$ spans an extreme ray in the convex cone $\mathcal{V}(\eta_{i(j)})$ of all transverse measures on $\eta_{i(j)}$.

Let $\mathcal{C}(S)$ be the *curve graph* of S and let $\Psi : \mathcal{TT} \to \mathcal{C}(S)$ be any map which associates to a complete train track $\sigma \in \mathcal{TT}$ a vertex cycle of σ. It was shown in [H06a] that there is a number $p > 1$ such that the image under Ψ of every splitting and shifting sequence $\{\sigma_i\} \subset \mathcal{TT}$ is an *unparametrized p-quasi-geodesic* in $\mathcal{C}(S)$. This means that there is an nondecreasing map $s : \mathbb{N} \to \mathbb{N}$ such that the assignment $i \to \Psi(\tau_{s(i)}) \in \mathcal{C}(S)$ is a uniform quasi-geodesic. Since the curve graph is hyperbolic and since the distance in $\mathcal{C}(S)$ between the curves c and $c_{i(j)}$ is uniformly bounded [MM99], this implies that the diameter of the image under Ψ of the splitting and shifting sequence $\{\tau_i\}$ is *finite*. However, the sequence $(x_i) \in \Omega$ is normal by assumption and hence the sequence $\{\tau_i\}$ contains subsequences whose images under the map F are unparametrized quasi-geodesics in $\mathcal{C}(S)$ with endpoints of arbitrarily large distance. This is a contradiction and implies that the support of λ fills up S as claimed. □

In other words, the set of normal points in Ω is contained in the set $\mathcal{U}$ of uniquely ergodic points. Since normal points are dense in Ω, the same is true for uniquely ergodic points.

Recall from Section 2 that for every $\tau \in \mathcal{TT}$ the cone $\mathcal{V}^*(\tau)$ of all tangential measures on τ can be identified with the space of all measured geodesic laminations which hit τ efficiently. If $\tau' \in \mathcal{TT}$ is obtained from $\tau \in \mathcal{TT}$ by a single split at a large branch e and if C is the matrix which describes the transformation $\mathcal{V}(\tau') \to \mathcal{V}(\tau)$ then there is a natural transformation $\mathcal{V}^*(\tau) \to \mathcal{V}^*(\tau')$ given by the transposed matrix C^t. This transformation maps the tangential measure on τ determined by a measured geodesic lamination ν hitting τ efficiently to the tangential measure on τ' defined by the same lamination (which hits τ' efficiently as well, compare [PH92]). Denote by $\mathcal{PE}(\tau)$ the space of projective measured geodesic laminations which hit τ efficiently. As in the proof of Lemma 5.3 we observe.

Lemma 5.4. *Let $(x_i) \in \Omega$ be normal and let $(\tau_i) \subset \mathcal{NT}$ be a numbered splitting and shifting sequence which realizes (x_i); then $\cap_i \mathcal{PE}(\tau_i)$ consists of a single uniquely ergodic projective measured geodesic lamination which fills up S.*

We call the sequence $(x_i) \in \Omega$ *doubly uniquely ergodic* if (x_i) is uniquely ergodic as defined above and if moreover for one (and hence every) numbered splitting and shifting sequence $(\tau_i) \in \mathcal{NT}$ which realizes (x_i) the intersection $\cap_{i<0}\mathcal{PE}(\tau_i)$ consists of a unique point. By Lemma 5.3 and Lemma 5.4, every normal sequence is doubly uniquely ergodic and hence the Borel set $\mathcal{DU} \subset \Omega$ of all doubly uniquely ergodic sequences $(x_i) \in \Omega$ is dense. Moreover, for each such sequence $(x_i) \in \mathcal{DU}$ and every numbered splitting and shifting sequence $(\tau_i) \subset \mathcal{NT}$ which realizes (x_i) there is a unique pair (λ^+, λ^-) of uniquely ergodic measured geodesic laminations which fill up S and which satisfy the following additional requirements.

(1) λ^+ is carried by each of the train tracks τ_i and the total mass disposed by λ^+ on the large branches of τ_0 equals one.
(2) λ^- hits each of the train tracks τ_i efficiently.
(3) $i(\lambda^+, \lambda^-) = 1$.

In particular, by equivariance under the action of the mapping class group, every sequence $(x_i) \in \mathcal{DU}$ determines a quadratic differential of total area one over a point in moduli space.

Let $\mathcal{UQ} \subset \mathcal{Q}(S)$ be the set of all area one quadratic differentials whose horizontal and vertical measure foliations are both uniquely ergodic and fill up S. Then $\mathcal{UQ}$ is a Φ^t-invariant Borel subset of $\mathcal{Q}(S)$.

A *suspension* for the shift T on the subspace $\mathcal{DU} \subset \Omega$ of all doubly uniquely ergodic sequences in the phase space Ω with roof function $\rho : \mathcal{DU} \to (0,\infty)$ is the space $X = \{(x_i) \times [0, \rho(x_i)] \mid (x_i) \in \mathcal{DU}\}/\sim$ where the equivalence relation $\sim$ identifies the point $((x_i), \rho(x_i))$ with the point $(T(x_i), 0)$. There is a natural flow Θ^t on X defined by $\Theta^t(x,s) = (T^j x, \tilde{s})$ (for $t > 0$) where $j \geq 0$ is such that $0 \leq \tilde{s} = s - \sum_{i=0}^{j-1} \rho(T^i x) \leq \rho(T^j x)$. A *semi-conjugacy* of (X, Θ^t) into a flow space (Y, Φ^t) is a (continuous) map $\Xi : X \to Y$ such that $\Phi^t \Xi(x) = \Xi(\Theta^t x)$ for all $x \in X$ and all $t \in \mathbb{R}$. We call a semi-conjugacy Ξ *countable-to-one* if the preimage of any point is at most countable. As an immediate consequence of Lemma 5.2 we obtain.

Corollary 5.5. *There is a countable-to-one semi-conjugacy Ξ of the Borel suspension X for the shift T on $\mathcal{DU}$ with roof function ρ to the Teichmüller geodesic flow. The image of Ξ is the Φ^t-invariant subset $\mathcal{UQ} \subset \mathcal{Q}(S)$.*

Proof. The existence of a countable-to-one semi-conjugacy Ξ as stated in the corollary is clear from the above considerations. We only have to show that this semi-conjugacy is continuous and that its is image equals precisely the set $\mathcal{UQ}$.

Continuity of Ξ follows as in the proof of Lemma 5.1. To show that the image of Ξ is all of $\mathcal{UQ}$ let (λ^+, λ^-) be any pair of distinct uniquely ergodic geodesic laminations with $i(\lambda^+, \lambda^-) = 1$. Then every leaf of λ^+ intersects every leaf of λ^- transversely. By the results from Section 2 of [H06b], there is a complete train track τ which carries λ^+ and which hits λ^- efficiently. For a suitable numbering of the branches of τ and up to modifying τ by a sequence of shifts the results of Section 4 show that we can choose our train track in such a way that it is contained in the set $\mathcal{E}$. By construction, this means that there is an infinite full splitting and shifting sequence $(\tau_i) \subset \mathcal{NT}$ such that the intersection $\cap_i \mathcal{PM}(\tau_i)$ consists of a unique point which is just the class of λ^+ and that the intersection $\cap_i \mathcal{EM}(\tau_i)$ consists of a unique point which is just the class of λ^-. This shows that our map Ξ maps $\mathcal{DU}$ onto $\mathcal{UQ}$. $\square$

A quadratic differential $q \in \mathcal{Q}^1(S)$ of area one defines a marked piecewise euclidean metric on our surface S. Every essential closed curve γ can be represented by a geodesic with respect to this metric. If λ^+, λ^- are the horizontal and vertical measured geodesic laminations for q then the length of γ with respect to the metric defined by q is contained in the interval $[\max\{i(\gamma, \lambda^+), i(\gamma, \lambda^-)\}, i(\gamma, \lambda^+) + i(\gamma, \lambda^-)]$.

Recall that a point in Teichmüller space can be viewed as a (marked) hyperbolic metric on S and that there is a number $\chi_0 > 0$ only depending on the topological type of our surface such that for every $x \in \mathcal{T}(S)$ there is a pants decomposition for S which consists of pants curves of hyperbolic length at most χ_0. Rafi [R05] showed that for every quadratic differential q of area one on the Riemann surface

corresponding to x the q-lengths of the pants curves of this short pants decomposition (i.e. the lengths of the q-geodesic representatives) are bounded from above by a universal constant $b > 0$. For every $\epsilon > 0$ there is a constant $c(\epsilon) > 0$ such that if x is contained in the *ϵ-thick part $\mathcal{T}(S)_\epsilon subset \mathcal{T}(S)$* of all marked hyperbolic metrics without closed geodesics of length less than ϵ then the hyperbolic metric and the q-metric are $c(\epsilon)$-bilipschitz equivalent (compare e.g. [CRS06] for this well known fact). However, for points in the thin part of Teichmüller space there may be curves which are short in the q-metric but which are long in the hyperbolic metric.

The next lemma gives some information on the curves which become short for the quadratic differential metric along a Teichmüller geodesic defined by a quadratic differential $q \in \mathcal{UQ}$. For its formulation, recall from [MM99] the definition of a *vertex cycle* for a train track τ. Such a vertex cycle is a simple closed curve c which is carried by τ and such that the transverse measure defined by c spans an extreme ray in the convex polygon of all transverse measures on τ. For simplicity, we identify Ω with a subset of our suspension space. We have.

Lemma 5.6. *Let $(x_i) \in \mathcal{DU}$ and let $(\tau_i) \subset \mathcal{NT}$ be a full numbered splitting and shifting sequence which realizes (x_i). Let $q = \Xi(x_i)$ and let $\tilde{q} \in \mathcal{Q}^1(S)$ be a lift of q defined by (τ_i). Then there is a vertex cycle on τ_0 of uniformly bounded $\tilde{q}$-length.*

Proof. Let $(x_i) \in \mathcal{DU}$ and let $(\tau_i) \subset \mathcal{NT}$ be a full numbered splitting and shifting sequence which realizes (x_i). Let (λ^+, λ^-) be the pair of transverse measured geodesic laminations which are defined by the lift $\tilde{q}$ of $\Xi(x_i)$ which is determined by (τ_i). By construction, the maximal weight that is disposed by $\lambda^+ \in \mathcal{V}(\tau_0)$ on a large branch of τ_0 is one and hence the total weight of the transverse measure on τ_0 defined by λ^+ is uniformly bounded. Thus by Corollary 2.3 of [H06a], the intersection number between λ^+ and any vertex cycle of τ_0 is uniformly bounded.

By the results of [MM99], the number of vertex cycles for a complete train track τ is uniformly bounded. Since the space $\mathcal{V}(\tau)$ of transverse measures on τ is spanned by the vertex cycles, there is a decomposition $\lambda^+ = \sum_i a_i c_i$ where $a_i \geq 0$ and c_i are the vertex cycles of τ_0. Now the number of vertex cycles is uniformly bounded and hence we have $a_i > \epsilon$ for a universal number $\epsilon > 0$ and at least one i. Then the weight that the tangential measure λ^- disposes on any branch b contained in the image of c_i is bounded from above by $1/\epsilon$. Since a vertex cycle runs through every branch of τ_0 at most twice [H06a], the intersection $i(\lambda^-, c_i)$ is bounded from above by twice the total weight of the measure λ^- on the branches of τ_0 in the image of c_i. Now $\sum_{b\in\tau} \lambda^+(b)\lambda^-(b) = 1$ and hence by our choice of c_i this total weight is uniformly bounded. But this means that $i(\lambda^+, c_i) + i(\lambda^-, c_i)$ is uniformly bounded and shows the lemma. □

Now let $q \in \mathcal{Q}(S)$ be a periodic point for the Teichmüller geodesic flow. Then $q \in \mathcal{UQ}(S)$ as is well known and therefore there is some $(x_i) \in \mathcal{DU}$ such that the orbit of the suspension flow through q is mapped by the semi-conjugacy Ξ onto the Φ^t-orbit of q. Since our semi-conjugacy Ξ is countable-to-one this orbit of the suspension flow need not be closed. However we have.

Lemma 5.7. *Let $(x_i) \in \mathcal{DU}$ be a point which is mapped by Ξ to a periodic point of the Teichmüller flow. Then the closure of the set $\{T^j(x_i) \mid j \in \mathbb{Z}\}$ is contained in $\mathcal{DU}$.*

Proof. The lemma is a consequence of a general observation about the piecewise euclidean metric defined by a quadratic differential $q \in \mathcal{DU}$ and its relation to the preimages of q in Ω. For this let $(x_i) \in \mathcal{DU}$ and let $q = \Xi(x_i) \in \mathcal{UQ}$. Let $(\tau_i) \subset \mathcal{NT}$ be a numbered splitting and shifting sequence which realizes (x_i) and let $\tilde{q} \in \mathcal{Q}^1(S)$ be the lift of q defined by (τ_i). Let λ^+, λ^- be the measured geodesic laminations defined by the quadratic differential $\tilde{q}$. Then the maximal weight that λ^+ disposes on a large branch of τ_0 equals one. By Lemma 5.2 we know that there is a number $p > 0$ such that $\sum_{j=0}^{p-1} \rho(T^j(x_i)) > 0$. We claim that there is a constant $k > 0$ only depending on q such that

$$\sum_{j=0}^{k-1} \rho(T^j(x_i)) \geq 1.$$

For a small number $\epsilon < 1/100$, assume that $\sum_{j=0}^{k-1} \rho(T^j(x_i)) < \epsilon$ for large k. Let $\sigma < \tau_0$ be the subgraph of τ_0 of all branches whose weight with respect to the measure μ defined by λ^+ is at least $e^{-\epsilon}$. It follows from the considerations in the proof of Lemma 5.2 that σ carries a simple closed curve c. Moreover, the μ-weight of every branch $b \in \tau_0$ which is incident on a switch in c and which is not contained in c is close to zero. By the considerations in Lemma 2.5 of [H06a], this implies that $i(\lambda^+, c)$ is very small (depending on ϵ).

Now if ν is the tangential measure on τ_0 defined by the vertical measured geodesic lamination of q then $\sum_b \mu(b)\nu(b) = 1$ and therefore

$$\sum_{b \in c} \nu(b) = i(\lambda^-, c)$$

is uniformly bounded. As a consequence, the q-length of c is uniformly bounded, and $\delta = i(\lambda^+, c)i(\lambda^-, c)$ is very small. Since the measured geodesic laminations λ^+, λ^- fill up S there is a unique point $t \in \mathbb{R}$ such that c is *balanced* with respect to $\Phi^t q$. This means that the intersection numbers $e^t i(\lambda^+, c), e^{-t} i(\lambda^-, c)$ conicide and are not smaller than half of the $\Phi^t q$-length of c. In other words, if $\sum_{j=0}^{k-1} \rho(T^j(x_i))$ is small then the minimum over all $t \in \mathbb{R}$ of the $\Phi^t q$-lengths of c is very small. On the other hand, if the orbit of q under the Teichmüller geodesic flow is periodic then for sufficiently small $\epsilon > 0$ it does not intersect the subset of Teichmüller space of hyperbolic metrics whose systole is smaller than ϵ. But then the $\Phi^t q$-length of any simple closed curve on S is uniformly bounded from below independent of t. This shows our claim.

Let $\Psi : \mathcal{TT} \to \mathcal{C}(S)$ be a map which associates to a complete train track on S one of its vertex cycles. By our above consideration and the results from [H06a], the assignment $k \to \Psi(T^k(x_i))$ is an L-quasi-geodesic for some $L > 0$ depending only on q.

However, this means that there is a constant $\ell > 0$ such that for each j the sequence $\{T^{m+j}(x_i) \mid m = 0, \ldots, \ell - 1\}$ has the following property. Let again (τ_i)

be a full numbered splitting and shifting sequence which realizes (x_i). Then for each m, the distance between $\Psi(\tau_{m+\ell})$ and $\Psi(\tau_m)$ in $\mathcal{C}(S)$ is at least three. Now there are only finitely many full numbered splitting and shifting sequences of length ℓ with this property. In particular, by the considerations in the proof of Lemma 5.4, there is a universal constant $b > 0$ with the following property. Let $(\tau_i)_{0\leq m}$ be any such sequence and let ν be *any* measured geodesic lamination which is carried by τ_m and such that the maximal weight disposed by ν on a large branch of τ equals one. Then the maximal weight disposed by ν on a large branch of τ_0 is at least $1+\epsilon$. This then implies that the function ρ is bounded from below on $\{T^j(x_i) \mid j \in \mathbb{Z}\}$ by a positive constant and that moreover the assignment $j \to \Psi\tau_j$ is a uniform quasi-geodesic in $\mathcal{C}(S)$. As a consequence, the sequence $\{T^j(x_i)\}$ is contained in a compact subset of $\mathcal{DU}$. This shows the lemma. □

6. Bernoulli measures for the Teichmüller flow

Masur and Veech [M82, V82, V86] constructed a probability measure in the Lebesgue measure class on the space $\mathcal{Q}(S)$ of area one quadratic differentials over moduli space. This measure is invariant, ergodic and mixing under the Teichmüller flow Φ^t, and it gives full measure to the space of quadratic differentials whose horizontal and vertical measured geodesic laminations are uniquely ergodic and fill up S. Moreover, the measure is absolutely continuous with respect to the strong stable and the unstable foliation.

In this section we use the results from Section 5 to construct an uncountable family of Φ^t-invariant probability measures on $\mathcal{Q}(S)$ including the Lebesgue measure which give full measure to the set of quadratic differentials with uniquely ergodic horizontal and vertical measured geodesic laminations. These measures are ergodic and mixing, with exponential recurrence to a fixed compact set, and they are absolutely continuous with respect to the strong stable and the unstable foliation. This completes the proof of Theorem 1 from the introduction.

We begin with an easy observation on Φ^t-invariant probability measures on $\mathcal{Q}(S)$.

Lemma 6.1. *Let μ be a Φ^t-invariant Borel probability measure on $\mathcal{Q}(S)$. Then $\mu(\mathcal{Q}(S) - \mathcal{UQ}) = 0$.*

Proof. Since $\mathcal{UQ}$ is a Φ^t-invariant Borel subset of $\mathcal{Q}(S)$ it is enough to show that there is no Φ^t-invariant Borel probability measure μ on $\mathcal{Q}(S)$ with $\mu(\mathcal{UQ}) = 0$.

For this assume to the contrary that such a measure exists. Note that $\mathcal{Q}(S)$ can be represented as a countable union of compact sets and hence there is some compact subset K of $\mathcal{Q}(S)$ with $\mu(K) \geq 3/4$. Now by a result of Masur [M82], for every $q \in K - \mathcal{UQ}$ there is a number $t(q) > 0$ such that $\Phi^t(q) \notin K$ for every $t \geq t(q)$. The thus defined function on $K - \mathcal{UQ}$ can be chosen to be measurable. Then there is some $T > 0$ such that $\mu\{q \in K \mid t(q) \leq T\} \geq 1/2$. By invariance of μ under the Teichmüller flow we have $\mu(\Phi^T K) - K \geq 1/2$ and therefore $\mu(\mathcal{Q}(S) - K) \geq 1/2$ which is a contradiction. The lemma follows. □

Recall from Section 5 the construction of the semi-conjugacy Ξ from our subshift of finite type (Ω, T) with roof function ρ onto the Φ^t-invariant set $\mathcal{UQ}$. Since the roof function ρ on $\mathcal{DU}$ is uniformly bounded and essentially positive, every T-invariant Borel probability measure on Ω which gives full mass to $\mathcal{DU}$ defines a finite invariant measure for the ρ-suspension flow. The image of this measure under the semi-conjugacy Ξ is a finite Φ^t-invariant Borel measure on $\mathcal{Q}(S)$ which we may normalize to have total mass one. This means that the map Ξ induces a map Ξ_* from the space $\mathcal{M}_T(\Omega)$ of T-invariant Borel probability measures on Ω which give full mass to $\mathcal{DU}$ into the space $\mathcal{M}_{\rm inv}(\mathcal{Q})$ of Φ^t-invariant probability measures on $\mathcal{Q}(S)$. We equip both spaces with the weak*-topology. We have.

Lemma 6.2. *The map Ξ_* is continuous.*

Proof. Since Ω is a compact metrizable space, the space of all probability measures on Ω is compact. Thus we only have to show that whenever $\mu_i \to \mu$ in $\mathcal{M}_T(\Omega)$ then $\Xi(\mu_i) \to \Xi(\mu)$. But this just means that for every continuous function f on $\mathcal{Q}(S)$ we have $\int \Xi \circ f d\mu_i \to \int \Xi \circ f d\mu$ which is obvious since Ξ is continuous. $\square$

We also have.

Lemma 6.3. *The image of Ξ_* is dense.*

Proof. There is a one-to-one correspondence between the space of all locally finite $\mathcal{M}(S)$-invariant Borel measures on the product $\mathcal{PML} \times \mathcal{PML} - \Delta$ which give full measure to the pairs of distinct uniquely ergodic points and the space of all Φ^t-invariant finite Borel measures on $\mathcal{Q}(S)$. Now every closed Teichmüller geodesic gives rise to such a measure which is a sum of Dirac measures supported on the endpoints of all lifts of the Teichmüller geodesic to Teichmüller space. Here the pair of endpoint is the pair in $\mathcal{PML}\times\mathcal{PML}-\Delta$ which consists of the horizontal and the vertical projective measured geodesic lamination of a quadratic differential defining a lift of the closed Teichmüller geodesic to Teichmüller space. Since the pairs of endpoints of Teichmüller geodesics are known to be dense in $\mathcal{PML} \times \mathcal{PML} - \Delta$ the set of Φ^t-invariant Borel probability measures which can be approximated by weighted point masses on closed geodesics is dense in $\mathcal{M}_{\rm inv}(\mathcal{Q})$. As a consequence, it is enough to show that every measure $\nu \in \mathcal{M}_{inv}(\mathcal{Q})$ which is supported on a single closed geodesic is contained in the image of Ξ_*.

To see that this is indeed the case, let q be a periodic point for the Teichmüller flow. Up to replacing q by its image under the time-t-map of the flow for some small t, we may assume that there is some $(x_i) \in \mathcal{DU}$ with $\Xi(x_i) = q$. Denote as before by (X, Θ^t) the suspension flow defined by the roof function ρ. For $T > 0$ define

$$\mu_T = \frac{1}{T}\int_0^T \delta(\Theta^t(x_i))dt$$

where by abuse of notation we view (x_i) as a point in X. By Lemma 5.7, the supports of these measures are all contained in a fixed compact subset of $\mathcal{DU}$, viewed as a subset of X. Therefore there is a sequence T_i going to infinity such that the measures μ_{T_i} converge as $i \to \infty$ weakly to a Θ^t-invariant measure μ. However, it follows immediately from the discussion in the proof of Lemma 5.7 that

the support of μ is contained in the preimage of the closed orbit q under the map Ξ. Then $\Xi_*\mu$ is a Borel probability measure on $\mathcal{Q}(S)$ which is supported on the Φ^t-orbit of q. This shows the lemma. □

Let k be the cardinality of the set $\mathcal{E}$ of numbered combinatorial types as in Lemma 4.2 and let $P = (p_{ij})$ be any (k,k)-matrix with non-negative entries $p_{ij} \geq 0$ such that $p_{ij} > 0$ if and only if $a_{ij} > 0$ (where the matrix $A = (a_{ij})$ is defined in the paragraph preceding Lemma 4.3). We require that (p_{ij}) is *stochastic*, i.e. that $\sum_j p_{ij} = 1$ for all i. By Corollary 4.5 and the definitions, there is a number $\ell > 0$ such that $P^{(\ell)} > 0$. Thus by the Perron Frobenius theorem, there is a unique positive *probability vector* $p = (p_1, \dots, p_k)$ with $p_i > 0, \sum_i p_i = 1$ and such that $Pp = p$ by our normalization. Then the probability vector p together with the stochastic matrix P defines a Markov probability measure ν for the subshift (Ω, T) of finite type which is invariant under the shift and gives full measure to the set of normal points. The ν-mass of a cylinder $\{x_0 = i\}$ $(i \leq k)$ equals p_i, and the ν-mass of $\{x_0 = i, x_1 = j\}$ is $p_{ji}p_i$. Note that the identity $\sum_i p_{ji}p_i = p_j$ for all j is equivalent to invariance of ν under the shift. Since by Corollary 4.5 our shift (Ω, T) is topologically mixing, the shift (Ω, T, ν) is equivalent to a Bernoulli shift (see p.79 of [Mn87]). In particular, it is ergodic and strongly mixing, with exponential decay of correlations, and it gives full measure to the subset $\mathcal{R} \subset \Omega$ of normal points. We call a shift invariant probability measure on Ω of this form a *Bernoulli measure*.

The support of every Bernoulli measure μ on Ω is all of Ω, and μ is invariant and ergodic under the action of the shift T. The flow Θ^t on the Borel suspension X with roof function ρ constructed in Section 4 preserves the Borel measure $\tilde{\mu}$ given by $d\tilde{\mu} = d\mu \times dt$ where dt is the Lebesgue measure on the flow lines of our flow. Since by Lemma 5.2 the roof function ρ on $\mathcal{R}$ is bounded, the measure $\tilde{\mu}$ is finite and moreover ergodic under the action of the flow Θ^t. The push-forward $\Xi_*\tilde{\mu}$ of the measure $\tilde{\mu}$ by the semi-conjugacy Ξ defined in Corollary 5.5 is a finite Φ^t-invariant ergodic measure on $\mathcal{Q}(S)$ which gives full mass to the quadratic differentials with uniquely ergodic horizontal and vertical measured geodesic laminations which fill up S. We call its normalization to a probability measure on $\mathcal{Q}(S)$ a *Bernoulli measure* for the Teichmüller geodesic flow Φ^t. By construction, a Bernoulli measure for Φ^t is absolutely continuous with respect to the strong stable and the unstable foliation.

Proposition 6.4. *A Bernoulli measure for the Teichmüller geodesic flow is mixing.*

Proof. Let $\mathcal{UE}(S) \subset \mathcal{PML}$ be the Borel set of all projective measured geodesic laminations on S which are uniquely ergodic and whose support fills up S and denote by Δ the diagonal in $\mathcal{PML} \times \mathcal{PML} \supset \mathcal{UE}(S) \times \mathcal{UE}(S)$. The set $\mathcal{Q}^1_u$ of quadratic differentials in $\mathcal{Q}^1(S)$ with uniquely ergodic horizontal and vertical measured geodesic lamination which fill up S is (non-canonically) homeomorphic to $(\mathcal{UE}(S) \times \mathcal{UE}(S) - \Delta) \times \mathbb{R}$ by choosing a Borel map which associates to a pair of transverse projective measured geodesic laminations in $(\mathcal{UE}(S) \times \mathcal{UE}(S) - \Delta) \times \{0\}$ a point on the Teichmüller geodesic determined by the pair and extending this map in such a way that it commutes with the natural $\mathbb{R}$-actions. With this identification, the Teichmüller geodesic flow acts on $(\mathcal{UE}(S) \times \mathcal{UE}(S) - \Delta) \times \mathbb{R}$ via $\Phi^t(\lambda, \nu, s) = (\lambda, \nu, t + s)$. In particular, there is a natural orbit space projection $\Pi : \mathcal{Q}^1_u \to \mathcal{UE}(S) \times \mathcal{UE}(S) - \Delta$ which is equivariant with respect to the natural

action of the mapping class group on $\mathcal{Q}^1_u$ and the diagonal action of $\mathcal{M}(S)$ on $\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$.

Let μ be a Bernoulli measure for the Teichmüller geodesic flow. Then μ lifts to a Φ^t-invariant $\mathcal{M}(S)$-invariant Radon measure on $\mathcal{Q}^1(S)$, and this Radon measure disintegrates to a Radon measure $\hat\mu$ on $\mathcal{PML}\times\mathcal{PML}-\Delta$ which is invariant under the diagonal action of $\mathcal{M}(S)$ and which gives full mass to the invariant subset $\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$. Since μ is ergodic under the Teichmüller geodesic flow, the measure $\hat\mu$ is ergodic under the action of $\mathcal{M}(S)$. Assume to the contrary that the measure μ is not mixing. Then there is a continuous function φ on $\mathcal{Q}(S)$ with compact support and $\int\varphi d\mu=0$ so that $\varphi\circ\Phi^t$ does not converge weakly to zero as $t\to\infty$. We follow [B02] and conclude that there is a non-constant function ψ which is the almost sure limit of Cesaro averages of φ of the form $\frac{1}{k}\sum_{j=1}^k\varphi_{n_j},\frac{1}{k}\sum_{j=1}^k\varphi_{-n_j}$, i.e. for positive and negative times. Replacing ψ by the function $v\to\int_0^\epsilon\psi(\Phi^sv)ds/\epsilon$ for sufficiently small ϵ guarantees that there is a subset $E_0\subset\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$ of full $\hat\mu$-measure so that the lift $\tilde\psi$ to $\mathcal{Q}^1(S)$ of the resulting function is well defined and continuous on the lines in $\Pi^{-1}(E_0)$. The periods of the function $\tilde\psi$ define a measurable $\mathcal{M}(S)$-invariant map of E_0 into the set of closed subgroups of $\mathbb{R}$ which is constant by ergodicity. Since $\tilde\psi\not\equiv 0$ by assumption, the set $E_1\subset E_0$ where this group is of the form $a\mathbb{Z}$ for some $a\geq 0$ has full measure.

The Radon measure $\hat\mu$ on $\mathcal{PML}\times\mathcal{PML}-\Delta$ is absolutely continuous with respect to the product foliation of $\mathcal{PML}\times\mathcal{PML}$. This means that $\hat\mu$ is locally of the form $d\hat\mu=fd\hat\mu^+\times d\hat\mu^-$ where $\hat\mu^\pm$ is a measure on the space of projective measured geodesic laminations whose measure class is invariant under the action of $\mathcal{M}(S)$ and where f is a positive measurable function. Since the action of $\mathcal{M}(S)$ on $\mathcal{PML}$ is minimal, the measures μ^+,μ^- are of full support. Now the horizontal measured geodesic lamination of a typical point for the measure μ on $\mathcal{Q}(S)$ is uniquely ergodic and fills up S and hence if v,w are typical points for μ on the same strong stable manifold in $\mathcal{Q}(S)$ then $\lim_{t\to\infty}d(\Phi^tv,\Phi^tw)=0$ for a suitable choice of a distance function d on $\mathcal{Q}(S)$ [M80]. From continuity of the function φ and absolute continuity of μ with respect to the strong stable and the unstable manifolds we deduce that $\psi(v)=\psi(w)$. Similarly we argue for the strong unstable manifolds (see the nice exposition of this argument in [B02]). As a consequence, there is an $\mathcal{M}(S)$-invariant subset $E_2\subset E_1$ of full $\hat\mu$-mass such that for every $\zeta\in E_2$ and every $q\in\Pi^{-1}(\zeta)$ the function $\tilde\psi$ is constant almost everywhere along the strong stable manifold $W^{ss}(q)$ and along the strong unstable manifold $W^{su}(q)$.

As in [B02] we define

$$\begin{aligned}E^+&=\{x\in\mathcal{UE}(S)\mid(x,y')\in E_2\text{ for }\hat\mu^-\text{ a.e. }y'\},\\ E^-&=\{y\in\mathcal{UE}(S)\mid(x',y)\in E_2\text{ for }\hat\mu^+\text{ a.e. }x'\}.\end{aligned}\tag{7}$$

By absolute continuity, the set $E^+\times E^-$ has full measure with respect to $\hat\mu$.

Choose a density point $(u^+,u^-)\in E^+\times E^-$ for $\hat\mu$ and let V^+,V^- be small disjoint neighborhoods of u^+,u^- in $\mathcal{PML}$ with the property that for every $x\in V^+$ and every $y\in V^-$ the projective measured geodesic laminations x,y together fill up S. Such neighborhoods are well known to exist. Then every pair $(x,y)\in$

$V^+ \times V^-$ determines a unique Teichmüller geodesic. Define a *cross ratio* function $\sigma : V^+ \times V^- \to \mathbb{R}$ as follows. Choose a quadratic differential $q \in \mathcal{Q}^1(S)$ whose horizontal measured geodesic lamination λ^+ is in the class u^+ and whose vertical measured geodesic lamination λ^- is in the class u^-. For $y \in V^-$, the pair (u^+, y) defines a Teichmüller geodesic whose corresponding one-parameter family of quadratic differentials intersects $W^{ss}(q)$ in a unique point $\alpha_1(x,y)$. Similarly, the family of quadratic differentials defined by the pair (x,y) intersects $W^{su}(\alpha_1(x,y))$ in a unique point $\alpha_2(x,y)$, the family of quadratic differentials defined by the pair (x,u^-) intersects $W^{ss}(\alpha_2(x,y))$ in a unique point $\alpha_3(x,y)$, and finally the family of quadratic differentials defined by (u^+,u^-) intersect $W^{su}(\alpha_3(x,y))$ in a unique point $\Phi^{\sigma(x,y)}(q)$. The value $\sigma(x,y) \in \mathbb{R}$ does not depend on the choice of the quadratic differential q defining the pair (u^+,u^-). The resulting function $\sigma : V^+ \times V^- \to \mathbb{R}$ is continuous and satisfies $\sigma(u^+,u^-) = 0$.

Recall that there is a natural action of the group $SL(2,\mathbb{R})$ on $\mathcal{Q}^1(S)$ where the diagonal subgroup of $SL(2,\mathbb{R})$ acts as the Teichmüller geodesic flow. Each $SL(2,\mathbb{R})$-orbit can naturally and equivariantly be identified with the unit tangent bundle of the hyperbolic plane together with its usual action by isometries. In particular, for every $z \in \mathcal{Q}^1(S)$ the orbit through z of the unipotent subgroup of $SL(2,\mathbb{R})$ of all upper triangular matrices of trace two is an embedded line in $W^{ss}(z)$, and the orbit through z of the unipotent subgroup of all lower triangular matrices of trace two is contained in $W^{su}(z)$. Thus the $SL(2,\mathbb{R})$-orbit through the above quadratic differential q defines two embedded line segments $I^+ \subset V^+, I^- \subset V^-$ containing u^+, u^- in their interior so that the restriction of the function σ to $I^+ \times I^-$ coincides with the restriction of the usual dynamical cross ratio on the space $S^1 \times S^1 - \Delta$ of oriented geodesics in the hyperbolic plane. In particular, the restriction of the function σ to $I^+ \times I^-$ is not constant in no neighborhood of (u^+,u^-). On the other hand, if we write $U^+ = V^+ \cap E^+, U^- = V^- \cap E^-$ then U^+, U^- are dense in V^+, V^-. Since our function σ is continuous and not constant in any neighborhood of (u^+,u^-) we conclude that for every $\epsilon > 0$ there are points $x \in U^+, y \in U^-$ such that $\sigma(x,y) \in (0,\epsilon)$. However, by our choice of E^+, E^-, the function $\tilde\psi$ is constant along the manifolds $W^{ss}(q), W^{su}(\alpha_1(x,y)), W^{ss}(\alpha_2(x,y))$ and $W^{su}(\alpha_3(x,y))$ and therefore $\tilde\psi(\Phi^{\sigma(x,y)}q) = \tilde\psi(q)$. In other words, the function $\tilde\psi$ has arbitrarily small periods which is a contradiction to our assumption that the set of periods of $\tilde\psi$ equals $a\mathbb{Z}$ for some $a \geq 0$. This completes the proof of our proposition. □

We also obtain a control of the return time to a suitably chosen compact subset of $\mathcal{Q}(S)$ for a flow line of the Teichmüller geodesic flow Φ^t on $\mathcal{Q}(S)$ which is typical for a Bernoulli measure μ on $\mathcal{Q}(S)$. The following observation is a version of Theorem 2.15 of [AGY05] for all Bernoulli measures (see also [A05]).

Lemma 6.5. *There is a compact subset K of $\mathcal{Q}(S)$ and for every Bernoulli measure μ for the Teichmüller geodesic flow Φ^t there is a number $\epsilon = \epsilon(\mu) > 0$ and a constant $C > 0$ such that*

$$\mu\{q \mid \text{for all}\, s \in [0,t], \Phi^s q \notin K\} \leq Ce^{-\epsilon t}.$$

Proof. Let $(y_i)_{0 \leq i \leq k}$ be an admissible sequence with the additional property that there is some $\ell < k$ such that the sequences $(y_i)_{0 \leq i \leq \ell}$ and $(y_i)_{\ell \leq i \leq k}$ are both tight. By Lemma 5.2 of [H06b] there is a compact subset K of $\mathcal{Q}(S)$ such that for

every numbered splitting and shifting sequence $\{\tau_i\}_{0\leq i\leq k}$ which realizes $(y_i)_{0\leq i\leq k}$ the following holds. Let $\lambda \in \mathcal{ML}$ be a measured geodesic lamination which is carried by τ_k and which defines a transverse measure on τ_0 for which the sum of the masses of the large branches of τ_0 equals one. Let $\nu \in \mathcal{ML}$ be a measured geodesic lamination which hits τ_0 efficiently and such that $i(\lambda,\nu)=1$. Then the quadratic differential $q(\lambda,\nu)$ with horizontal measured geodesic lamination λ and vertical measured geodesic lamination ν is contained in the lift of K to $\mathcal{Q}^1(S)$.

Let μ be any Bernoulli measure on the shift space (Ω, T). Then μ is exponentially mixing and hence there are constants $c_0>0, \epsilon>0$ such that $\mu\{(x_i)\in\Omega \mid (y_i)_{0\leq i\leq k}\not\subset (x_j)_{0\leq j\leq m}\}\leq c_0 e^{-\epsilon m}$. The corollary now follows from this observation and the fact that the roof function ρ on $\mathcal{R}$ which defines our Borel suspension which is semi-conjugate to the Teichmüller flow is uniformly bounded. □

We conclude this section with a description of the $\mathcal{M}(S)$-invariant Φ^t-invariant measure λ in the Lebesgue measure class on the unit cotangent bundle $\mathcal{Q}^1(S)$ of Teichmüller space which is invariant under the natural action of the group $SL(2,\mathbb{R})$ and which projects to a probability measure on $\mathcal{Q}(S)$. This measure induces a $\mathcal{M}(S)$-invariant Radon measure λ_0 on $\mathcal{PML}\times\mathcal{PML}-\Delta$.

Using the notations from Section 4, let $\mathcal{E}$ be the collection of all numbered combinatorial types as in Lemma 4.2. For every $j\in\mathcal{E}$ choose a numbered train track $\tau(j)$ of combinatorial type j. Let $W^+(j)\subset\mathcal{PML}$ be the space of all projective measured geodesic laminations which are carried by $\tau(j)$ and let $W^-(j)$ be the space of all projective measured geodesic laminations which hit $\tau(j)$ efficiently. Note that $W^+(j), W^-(j)$ are closed disjoint subsets of $\mathcal{PML}$ with dense interior.

Define a *full split* of $\tau(j)$ to be a complete train track σ which can be obtained from $\tau(j)$ by splitting $\tau(j)$ at each large branch precisely once. Since splits at different large branches of $\tau(j)$ commute, the numbering of the branches of $\tau(j)$ induces a numbering of the branches of σ. Thus the train tracks $\sigma^1,\dots,\sigma^\ell$ obtained from $\tau(j)$ by a full split are numbered. We obtain a decomposition $W^+(j)=\cup_{i=1}^{\ell}W^i$ into Borel sets W^i where $W^i\subset W^+(j)$ consists of the set of all projective measured geodesic laminations which are carried by σ^i $(i=1,\dots,\ell)$. For $i\neq j$ the intersection $W^i\cap W^j$ is contained in a hyperplane of $\mathcal{PML}$ (with respect to the natural piecewise linear structure which defines the Lebesgue measure class) and hence it has vanishing Lebesgue measure. If $k(i)\in\mathcal{E}$ is the combinatorial type of the numbered train track σ^i then define

$$p_{jk(i)}=\lambda_0(W^i\times W^-(j))/\lambda_0(W^+(j)\times W^-(j)) \tag{8}$$

and define $p_{ju}=0$ for $u\not\in\{k(i)\mid i\}$. Note that we have $p_{js}<1$ for all s and $\sum_s p_{js}=1$. By invariance of the measure λ_0 under the action of the mapping class group, the probabilities $p_{jk(i)}$ only depend on the combinatorial type $j\in\mathcal{E}$ and the choice of a representative in the shift equivalence class of j.

Consider the random walk on the space $\mathcal{Z}$ of all numbered classes defined by the $\mathcal{M}(S)$-invariant transition probabilities (p_{ij}). It follows from our above consideration that the exit boundary of this random walk can be viewed as a probability measure on $\mathcal{PML}$ in the Lebesgue measure class. By invariance under the action of the mapping class group, the shift invariant measure μ on the subshift

(Ω_0, T) of finite type with alphabet $\mathcal{E}$ and transition matrix defined by the above full splits is contained in the Lebesgue measure class. Therefore the measure on $\mathcal{PML}\times\mathcal{PML}-\Delta$ defined by the biinfinite random walk with the above transition probabilities coincides with λ_0 up to scale. We use this observation to obtain a new (and simpler) proof of Theorem 2.15 of [AGY05] and of the main theorem of [A05].

Proposition 6.6. *Let λ be the Φ^t-invariant probability measure on $\mathcal{Q}(S)$ in the Lebesgue measure class. There is a compact subset K of $\mathcal{Q}(S)$ and a number $\epsilon > 0$ such that $\lambda\{q \mid \Phi^s q \notin K \text{ for every } s \in [0,t]\} \leq e^{-\epsilon t}/\epsilon$.*

Proof. Even though a priori it is not clear whether our subshift of finite type is topologically mixing (or even transitive), we can apply the same consideration as above. Namely, since set of uniquely ergodic minimal and maximal measured geodesic laminations has full Lebesgue measure, there is a finite sequence $(y_i)_{0\leq i\leq k} \subset \mathcal{E}$ with $p_{y_i y_{i+1}} > 0$ for all i and there is a number $\ell > 0$ such that for every full splitting and shifting sequence $\{\tau_i\}_{0\leq i\leq k}$ which realizes (y_i) in the above sense, the sequences $\{\tau_i\}_{0\leq i\leq \ell}$ and $\{\tau_i\}_{\ell\leq i\leq k}$ are tight in the sense described before Lemma 4.6. As in the proof of Lemma 5.2, from this observation the proposition follows (compare also the discussion in [K85]). ☐

7. The critical exponent of $\mathcal{M}(S)$

In this section we derive the first part of Theorem C from the introduction. Namely, we show that the critical exponent of the mapping class group equals $6g - 6 + 2m$. For the proof of this result we continue to use the assumptions and notations from Section 2-6. In particular, we denote by d the Teichmüller distance on $\mathcal{T}(S)$.

Fix a point $x \in \mathcal{T}(S)$. The Poincaré series at x with exponent $\alpha > 0$ is defined to be the series

$$\sum_{g\in\mathcal{M}(S)} e^{-\alpha d(x,gx)}. \tag{9}$$

The critical exponent of $\mathcal{M}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series with exponent α converges. Note that this critical exponent does not depend on the choice of x. We first give a lower bound for this critical exponent.

Lemma 7.1. *The Poincaré series diverges at the exponent $h = 6g - 6 + 2m$.*

Proof. Let $q_0 \in \mathcal{Q}(S)$ be a forward recurrent point for the Teichmüller geodesic flow and let $q_1 \in \mathcal{Q}^1(S)$ be a lift of q_0. Denoting again by $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ the canonical projection, we may assume that $Pq_1 = x$ is not fixed by any nontrivial element of $\mathcal{M}(S)$. In particular, there is a number $r > 0$ such that the images under the elements of $\mathcal{M}(S)$ of the ball of radius $2r$ about x are pairwise disjoint.

Using the notations from Proposition 3.3 of [H07] and its proof, let $\mathcal{FML}$ be the set of all projective measured geodesic laminations which contain a minimal sublamination which fills up S. Let $\partial\mathcal{C}(S)$ be the Gromov boundary of the curve graph.. Then there is a natural continuous, surjective and closed map $F : \mathcal{FML} \to$

$\partial\mathcal{C}(S)$. For $q \in \mathcal{Q}^1(S)$ let λ_q be the image under the map $F \circ \pi : \pi^{-1}\mathcal{FML} \to \partial\mathcal{C}(S)$ of the Lebesgue measure λ^{su} on the strong unstable manifold $W^{su}(q)$. Let $A \subset \mathcal{Q}^1(S)$ be the preimage under π of the subset $\mathcal{RML}(q_0)$ of all recurrent projective measured geodesic laminations determined by q_0. By this we mean that for $\nu \in \mathcal{RML}$ the point q_0 is contained in the ω-limit set of the projection to $\mathcal{Q}(S)$ of any flow line of the Teichmüller flow with horizontal lamination contained in the class ν. For $q \in A$ and $\chi > 0$ denote by $D(q,\chi)$ the closed δ_{Pq}-ball of radius χ about $F\pi(q) \in \partial\mathcal{C}(S)$ We showed in Proposition 3.3 of [H07] that there is a number $\chi > 0$ and there is an open neighborhood U of q_1 in $\mathcal{Q}^1(S)$ with the following properties.

(1) The projection of U to $\mathcal{T}(S)$ is contained in the ball of radius r about Pq_1.
(2) There is a number $a > 0$ such that for all $q, z \in U$ we have $\lambda_q(D(z,\chi)) \le a$.
(3) The family $\mathcal{V}$ of all pairs $(F\pi(q), gD(q_1,\chi)) \mid q \in W^{su}(q_1, \Phi^t q \in gU$ for some $t > 0\}$ is a Vitali relation for λ_{q_1}.

By the proof of Proposition 3.3 of [H07], there is an open relative compact subset W of $W^{su}(q_1)$ with the property that $V = F(\pi(W) \cap \mathcal{FML})$ is an open neighborhood of $F\pi(q_1)$ in $\partial\mathcal{C}(S)$. Let $\alpha > 0$ be the diameter of the projection of W to $\mathcal{T}(S)$. By 3) above there is a countable set $G_1 = \{g_i \mid i > 0\} \subset \mathcal{M}(S)$ such that the sets $g_iD(q_1,\chi) = D(g_iq_1,\chi)$ $(g_i \in G_1)$ are pairwise disjoint, are contained in V and cover λ_{q_1}-almost all of V. Moreover, for each i there is some $p_i \in W \cap A$ and some $t_i > 0$ such that $\Phi^{t_i}p_i \in g_iU$. By 2) above we have $\lambda_{\Phi^{t_i}p_i}D(g_iq_1,\chi) \le a$, and 1) above implies that $d(x, g_ix) \ge t_i - d(x, Pp_i) - d(P\Phi^{t_i}p_i, g_ix) \ge t_i - \alpha - r$. From this together with the transformation rule for the measures λ^{su} under the action of the Teichmüller flow we obtain with $h = 6g - 6 + 2m$ that

$$b = \lambda_{q_1}(V) \le a\sum_i e^{-ht_i} \le c\sum_{g\in G_1} e^{-hd(x,gx)} \tag{10}$$

where $c = ae^{h(\alpha+r)}$ is a fixed constant.

Let $H_1 = \mathcal{M}(S) - G_1$. Since the closed sets $D(g_iq_1,\chi) \subset \partial\mathcal{C}(S)$ $(g_i \in G_1)$ are pairwise disjoint by assumption, every point $\xi \in V$ is contained in at most one of these sets. As a consequence, the family $\mathcal{V}_1 = \{(\xi, D(hq_1,\chi)) \in \mathcal{V} \mid h \in H_1, \xi \in D(hq_1,\chi)\}$ forms a covering relation for C which is fine at every point of C. Using again Proposition 3.3 of [H07] and its proof, this covering relation is a Vitali relation for the measure λ_{q_1}. In particular, we can find a countable subset $G_2 = \{h_i \mid i > 0\} \subset H_1$ $(i > 0)$ as above such that the sets $D(h_iq_1,\chi)$ are pairwise disjoint, are contained in V and cover λ_{q_1}-almost all of V. Summing over $G_1 \cup G_2$ yields that

$$\sum_{g\in G_1\cup G_2} e^{-hd(x,gx)} \ge 2b/c. \tag{11}$$

Inductively we conclude that $\sum_{g\in\mathcal{M}(S)} e^{-hd(x,gx)} \ge R$ for every $R > 0$. In other words, the Poincaré series diverges at the exponent h. $\square$

With the same method we obtain.

Lemma 7.2. *For every $\epsilon > 0$ the Poincaré series converges at $h + \epsilon$.*

Proof. For $x \in \mathcal{T}(S)$ let $\mathcal{Q}^1(S)_x \subset \mathcal{Q}^1(S)$ be the set of all unit area quadratic differentials which project to x. The sets $\mathcal{Q}^1(S)_x$ $(x \in \mathcal{T}(S))$ form a foliation of $\mathcal{Q}^1(S)$ of dimension $6g-7+2m$, and the same is true for the sets $\Phi^t\mathcal{Q}^1(S)_x$ for some fixed $t>0$.

Let $\{\lambda^x\}$ be the conformal density of dimension $6g-6+2m=h$ in the Lebesgue measure class. For $q \in \mathcal{Q}^1(S)$ and for $R>0$ write $\lambda_{q,R} = e^{hR}\lambda^{P\Phi^{-R}q}$.

Let $x \in \mathcal{T}(S)$ be a point which is not fixed by any nontrivial element of $\mathcal{M}(S)$. There is a number $r>0$ such that the images under the elements of $\mathcal{M}(S)$ of the balls of radius $4r$ about x are pairwise disjoint. For $q \in \mathcal{Q}^1(S)_x$ and $R>0$ let $B(q,R,r)$ be the set of all points $z \in \Phi^R\mathcal{Q}^1(S)_{P\Phi^{-R}q}$ which project into the closed ball of radius r about x. Using continuity and compactness, for sufficiently large R, say for all $R>R_0$, and for all $q \in \mathcal{Q}^1(S)_x$ the $\lambda_{q,R}$-mass of $\pi B(q,R,r)$ is bounded from below by a universal constant $b>0$ (compare the discussion in the proof of Proposition 3.3 of [H07]).

For an integer $k>0$ let $G(k) = \{g \in \mathcal{M}(S) \mid d(x,gx) \in [kr,(k+1)r)\}$. We claim that there is a constant $c>0$ such that for all $k \geq R_0/r$ the cardinality of the set $G(k)$ is bounded from above by ce^{hkr}. Then for every $\epsilon>0$ we have

$$\sum_{g\in\mathcal{M}(S)} e^{-(h+\epsilon)d(x,gx)} \leq \sum_{d(x,gx)\leq R_0} e^{-(h+\epsilon)d(x,gx)} + c\sum_{k\geq R_0/r} e^{hkr}e^{-(h+\epsilon)kr} < \infty \tag{12}$$

which shows our lemma.

To show our claim, let $k>R_0/r$, let $g,h \in G(k)$ and let $t(g)=d(x,gx), t(h)=d(x,hx)$. Let moreover $q,z \in \mathcal{Q}^1(S)_x$ be such that $P\Phi^{t(g)}q = gx, P\Phi^{t(h)}z = hx$. We claim that if $g \neq h$ then the sets $\pi B(\Phi^{t(q)}q,t(g),r)$ and $\pi B(\Phi^{t(h)}z,t(h),r)$ are disjoint. To see this assume otherwise. Since the restriction of the projection π to $\mathcal{Q}^1(S)_x$ is a homeomorphism, there is then a quadratic differential $u \in \mathcal{Q}^1(S)_x$ with $d(P\Phi^{t(g)}u,gx) \leq r, d(P\Phi^{t(h)}u,hx) \leq r$. Now $|t(g)-t(h)| \leq r$ and therefore we have $d(P\Phi^{t(g)}u,hx) \leq 2r$ and hence $d(gx,hx) \leq 3r$. By our choice of r, this implies that $g=h$.

The $\lambda_{\Phi^{t(g)}q,t(g)}$-mass of the set $\pi B(\Phi^{t(g)}q,t(g),r)$ is bounded from below by b and consequently by the definition of the measure $\lambda_{q,R}$ the λ^x-mass of $\pi B(\Phi^{t(g)}q,t(g),r)$ is bounded from below by $be^{-h(k+1)r}$. Since by our above considerations the sets $\pi B(\Phi^{t(q)}q,t(g),r)$ $(g \in G(k))$ are pairwise disjoint, their number does not exceed a universal constant times $e^{h(k+1)r}$. From this our above claim and hence the lemma is immediate. $\square$

As an immediate corollary of Lemma 7.1 and Lemma 7.2 we obtain.

Corollary 7.3. *The critical exponent of $\mathcal{M}(S)$ equals $6g-6+2m$.*

Proof. By Lemma 7.1, the Poincaré series of $\mathcal{M}(S)$ diverges at $h=6g-6+2m$ and hence the critical exponent of $\mathcal{M}(S)$ is not smaller than h. On the other hand, Lemma 7.2 shows that for every $\epsilon>0$ the Poincaré series converges at $h+\epsilon$

and hence the critical exponent is not larger that $h+\epsilon$. Together the corollary follows. ☐

Remark: Shortly after this work was completed, a preprint of Athreya, Bufetov, Eskin and Mirzakhani [ABEM06] appeared which contains precise asymptotics for the Poincaré series.

8. Counting closed Teichmüller geodesics

For $r>0$ let $n(r)$ be the number of periodic orbits for the Teichmüller flow on $\mathcal{Q}(S)$ of period at most r. Veech [V86] showed that there is a number $k>6g-6+2m$ such that $6g-6+2m\leq\liminf_{r\to\infty}\frac{1}{r}\log n(r)\leq\limsup_{r\to\infty}\frac{1}{r}\log n(r)\leq k$. Similarly, for $r>0$ and a compact set $K\subset\mathcal{Q}(S)$ let $n_K(r)$ be the cardinality of the set of all periodic orbits for the Teichmüller geodesic flow of period at most r which intersect K. By the results of [H06b], if $3g-3+m\geq 4$ then for every compact subset $K\subset\mathcal{Q}(S)$ and every sufficiently large r we have $n_K(r)<n(r)$. The next proposition shows the second part of Theorem C from the introduction (see also [Bu06] for a related result). For this we denote by $\mathrm{Mod}(S)=\mathcal{T}(S)/\mathcal{M}(S)$ the moduli space of S.

Proposition 8.1. *There is a compact subset K_0 of $\mathcal{Q}(S)$ such that*

$$\lim_{r\to\infty} n_K(r)=6g-6+2m$$

for every compact subset $K\supset K_0$ of $\mathcal{Q}(S)$.

Proof. Choose a framing F for S and let $\mathcal{P}$ be the collection of all train tracks in standard form for F. Let $\tau\in\mathcal{P}$ and recall that there is a constant $k>0$ such that every complete train track on S is at distance at most k from a train track in the collection $\{g\tau\mid g\in\mathcal{M}(S)\}$ in the *train track complex* of S (see [H06a] for the definition of the train track complex and its basic properties). Recall from Section 3 the definition of the map $\Lambda:\mathcal{TT}\to\mathcal{T}(S)$. There is a number $\ell>1$ such that $d(\Psi\tau,\Psi\sigma)\leq\ell d(\tau,\sigma)+\ell$ for all $\tau,\sigma\in\mathcal{TT}$ (where we denote by d the distance in the train track complex as well as the distance in $\mathcal{T}(S)$, see [H06a] and Lemma 3.4).

By Lemma 4.1 and the results from [H06b], there is a number $c>0$ and for every train track $\zeta\in\mathcal{TT}$ there are train tracks $\tau_\zeta\in\mathcal{P},\sigma\in\mathcal{TT}$ with the following properties.

(1) σ is contained in the $\mathcal{M}(S)$-orbit of τ_ζ.
(2) τ_ζ can be connected to σ by a splitting and shifting sequence.
(3) The distance in the train track complex between σ and ζ is bounded from above by c.

As a consequence, there is a number $L>0$ with the following property. Let $g\in\mathcal{M}(S)$; then there is some $h(g)\in\mathcal{M}(S)$ with $d(gx,h(g)x)\leq L$ and there is a train track $\tau(g)\in\mathcal{P}$ in standard form for F which can be connected to the train track $h(g)(\tau(g))$ by a splitting and shifting sequence.

Write $\mathcal{P}=\{\tau_1,\dots,\tau_k\}$ and for each $i\le k$ choose a tight splitting and shifting sequence connecting τ_i to a train track $g_i\tau_i$ for some $g_i\in\mathcal{M}(S)$; such a sequence exists by [H06b] and Section 4 (see [H06b] and the paragraph before Lemma 4.6 for the definition of a tight sequence). For $g\in\mathcal{M}(S)$ extend a splitting and shifting sequence connecting $\tau(g)=\tau_i$ to $h(g)(\tau(g))=h(g)(\tau_i)$ by the splitting and shifting sequence connecting $h(g)(\tau_i)$ to $h(g)g_i^2(\tau_i)$. Then the distance between gx and $h(g)g_i^2(x)$ is bounded from above by a universal constant. Moreover, by the results of [H06b], the element $h(g)g_i^2\in\mathcal{M}(S)$ is pseudo-Anosov and its axis passes through a fixed compact neighborhood of $\cup_{g\in\mathcal{M}(S)}gx$. In other words, there is a universal constant $C>0$ and for every $g\in\mathcal{M}(S)$ there is a pseudo-Anosov element $\rho(g)\in\mathcal{M}(S)$ with $d(\rho(g)x,gx)\le C$ whose axis projects to a closed geodesic γ in the moduli space with the following properties.

(1) The length of γ is contained in the interval $[d(x,gx)-C,d(x,gx)+C]$.
(2) γ passes through a fixed compact subset $\hat K_0$ of moduli space.

Let $K_0\subset\mathcal{Q}(S)$ be the preimage of $\hat K_0$ under the projection $\mathcal{Q}(S)\to\mathrm{Mod}(S)$. Since by Lemma 7.1 the Poincaré series diverges at the exponent $6g-6+2m$ and since moreover the number of elements $h\in\mathcal{M}(S)$ with $d(\rho(g)x,hx)\le C$ is bounded from above by a universal constant not depending on g, our above consideration implies that $\liminf_{r\to\infty}\frac{1}{r}\log n_{K_0}(r)\ge 6g-6+2m$.

We are left with showing that $\limsup_{r\to\infty}\frac{1}{r}\log n_K(r)\le 6g-6+2m$ for every compact subset K of $\mathcal{Q}(S)$. For this let $\hat K$ be any compact subset of $\mathrm{Mod}(S)$ with dense interior and let $K_1\subset\mathcal{T}(S)$ be a relatively compact fundamental domain for the action of $\mathcal{M}(S)$ on the lift $\tilde K$ of $\hat K$. Let D be the diameter of K_1. Let $x\in K_1$ be a point which is not fixed by any element of $\mathcal{M}(S)$. Let $g\in\mathcal{M}(S)$ be a pseudo-Anosov element whose axis projects to a closed geodesic γ in moduli space which intersects $\hat K$. Then there is a point $\tilde x\in K_1$ which lies on the axis of a conjugate of g which we denote again by g for simplicity. By the properties of an axis, the length $\ell(\gamma)$ of our closed geodesic γ equals $d(\tilde x,g\tilde x)$. On the other hand, we have $d(x,gx)\le d(\tilde x,g\tilde x)+2d(x,\tilde x)\le\ell(\gamma)+2D$ by the definition of D and the choice of $\tilde x$. Therefore, if we denote by $K\subset\mathcal{Q}(S)$ the preimage of $\hat K\subset\mathrm{Mod}(S)$ under the natural projection and if we define $N(r)$ for $r>0$ to be the number of all $g\in\mathcal{M}(S)$ with $d(x,gx)\le r$, then we have

$$\limsup_{r\to\infty}\frac{1}{r}\log n_K(r)\le\limsup_{r\to\infty}\frac{1}{r}N(r).\tag{13}$$

Since by Corollary 7.3 the critical exponent of the Poincaré series equals $6g-6+2m$, we conclude that $\limsup_{r\to\infty}\frac{1}{r}\log n_K(r)\le 6g-6+2m$ as claimed. ☐

For a point $x\in\mathrm{Mod}(S)$ define $s(x)$ to be the length of the shortest closed geodesic on the hyperbolic surface x. For a periodic geodesic $\gamma:[0,1]\to\mathrm{Mod}(S)$ write $s(\gamma)=\max\{s(\gamma(t))\mid t\in[0,1]\}$. We have.

Lemma 8.2. *There is a constant $\kappa>0$ such that the length of every closed geodesic in* $\mathrm{Mod}(S)$ *is at least* $(-\log s(\gamma)-\kappa)/(3g-3+m)$.

Proof. Let $\chi > 0$ be sufficiently small that for every complete hyperbolic metric h on S of finite volume and every closed geodesic c for the metric h of length at most χ, the length of every closed h-geodesic which intersects c nontrivially is bigger than 2χ. Via an appropriate choice of the constant $\kappa > 0$ in the statement of the lemma, it is enough to show the lemma for closed geodesics γ in moduli space with $s(\gamma) \leq \chi$.

Thus let $\gamma : [0, 1] \to \mathrm{Mod}(S)$ be such a closed geodesic parametrized proportional to arc length. Via reparametrization, assume that $s(\gamma) = s(\gamma(0))$. Let $\tilde{\gamma} : \mathbb{R} \to \mathcal{T}(S)$ be a lift of γ to $\mathcal{T}(S)$ which is the axis of the pseudo-Anosov element $g \in \mathcal{M}(S)$ and let c be an essential simple closed curve on $\gamma(0)$ of length $s(\gamma) \leq \chi$. We claim that the length of the geodesic arc $\tilde{\gamma}[0, 3g-3+m]$ is at least $\log\chi - \log s(\gamma)$. Namely, the function $\mathbb{R} \to (0, \infty)$ which associates to $t \in \mathbb{R}$ the length $\ell_{\tilde{\gamma}(t)}(c)$ of the geodesic representative of the simple closed curve c with respect to the hyperbolic metric $\tilde{\gamma}(t)$ is smooth, and the derivative of its logarithm does not exceed the length of the tangent vector of $\tilde{\gamma}$ (see [IT99]). Thus if the length of $\tilde{\gamma}[0, 3g - 3 + m]$ is smaller than $\log\chi - \log s(\gamma)$ then the length of the curve c with respect to the hyperbolic structure $\tilde{\gamma}(\ell)$ is smaller than χ for $\ell = 0, \dots, 3g - 3 + m$. By our choice of χ, for every ℓ there are at most $3g - 3 + m$ distinct simple closed geodesics of length at most χ for the hyperbolic metric $\tilde{\gamma}(\ell)$. However, since for each i the length of $g^i(c)$ on the hyperbolic surface $\tilde{\gamma}(i)$ equals $s(\gamma)$, the length of $g^i c$ on $\tilde{\gamma}(3g - 3 + m)$ does not exceed χ. As a consequence, there is some $i \in \{1, \dots, 3g - 3 + m\}$ such that the curves c and $g^i c$ coincide. This contradicts the fact that g is pseudo-Anosov. The lemma follows. ☐

As a corollary, we obtain a somewhat more precise version of the counting result of Veech [V86], with a new proof.

Corollary 8.3. $\limsup_{r\to\infty} \frac{1}{r} \log n(r) \leq (6g - 6 + 2m)(6g - 5 + 2m)$.

Proof. For $\alpha > 0$ let $\mathrm{Mod}(S)_\alpha$ be the set of all hyperbolic metrics on S up to isometry whose systole is at least α, i.e. $\mathrm{Mod}(S)_\alpha$ is the projection of the subset $\mathcal{T}(S)_\alpha$ defined in Section 3. By Proposition 8.1, there is a number $\epsilon > 0$ such that for every closed geodesic γ in moduli space of length at most r there is a point $z \in \mathrm{Mod}(S)_\epsilon$ whose distance to γ is not bigger than $(3g - 3 + m)r$. Namely, every $h \in \mathcal{T}(S)$ can be connected to $\mathcal{T}(S)_\epsilon$ with a geodesic of length at most $\log s(h) + C$ where $C > 0$ is a universal constant (see [M96]).

Let $R > 0$ be the diameter of $\mathrm{Mod}(S)_\epsilon$. For $x \in \mathcal{T}(S)_\epsilon$ there is a lift $\tilde{\gamma}$ of γ to $\mathcal{T}(S)$ which passes through a point y of distance at most $(3g - 3 + m)r + R$ to x. Let $g \in \mathcal{M}(S)$ be the pseudo-Anosov map preserving $\tilde{\gamma}$ which defines γ. We then have $d(gx, x) \leq r + 2(3g - 3 + m)r + 2R = (6g - 5 + 2m)r + 2R$ and hence our corollary follows from Lemma 7.2 and Proposition 8.1. ☐

To conclude, we reduce the counting problem for closed geodesics in Moduli space to a counting problem for pseudo-Anosov elements with small translation length in the curve graph. For this we fix again a point $x \in \mathcal{T}(S)$.

Lemma 8.4. *For every $\epsilon > 0$ there is a number $k(\epsilon) > 0$ with the following property. Let $x \in \mathcal{T}(S)$ and let γ be a closed geodesic in moduli space of length $\ell(\gamma)$. Let $C(\gamma) \subset \mathcal{M}(S)$ be the conjugacy class in $\mathcal{M}(S)$ defined by γ. If $g \in \mathcal{M}(S)$ is such that $d(x, gx) = \min\{h \in C(\gamma) \mid d(hx, x)\} \geq \ell(\gamma)/(1-\epsilon)$ then $d(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(gx)) \leq k(\epsilon)$.*

Proof. Let $\epsilon > 0$ and let γ be a closed geodesic in moduli space of length $\ell(\gamma)$. Let $\chi < 1$ be sufficiently small that no two simple closed geodesics with respect to a complete hyperbolic metric of finite volume on S of length at most χ can intersect. Let $K \subset \mathrm{Mod}(S)$ be the compact subset of moduli space of all hyperbolic metrics whose systole is at least χ.

Let $x \in \mathcal{T}(S)$ be a point which projects to a point in K and let $g \in C(\gamma)$ be such that $d(x, gx) = \min\{d(x, hx) \mid h \in C(\gamma)\}$. Assume that γ is parametrized in such a way that $s(\gamma) = s(\gamma(0))$. By the triangle inequality and the choice of g, there is a constant $a > 0$ not depending on γ and an element $g \in C(\gamma)$ such that with $R = -\log s(\gamma)$ we have $d(x, gx) \leq \ell(\gamma) + 2R + a$. Thus if $\ell(\gamma) \leq (1-\epsilon)d(gx, x)$ then we have $\ell(\gamma) \leq d(x, gx) \leq (2R + a)/\epsilon$.

Let c be a curve of length $s(\gamma)$ on $\gamma(0)$. With $\chi > 0$ as above, let $t_1 > 0$ be the smallest number such that the length of c on $\gamma(t)$ equals χ; then $t_1 \geq R + \log \chi$. By our choice of $s(\gamma)$, there is a curve c_1 on $\gamma(t_1)$ of length at most $s(\gamma)$, and this curve is necessarily disjoint from c. As a consequence, we have $d(c, c_1) = 1$ in the curve graph $\mathcal{C}(S)$. We repeat this consideration as follows. Let $t_2 > t_1$ be the smallest number such that the length of c_1 on $\gamma(t_2)$ is at least χ. As before, we have $t_2 \geq t_1 + R + \log \chi$. Thus after $k \leq 1/\epsilon$ steps we obtain a curve c_k on $g\gamma(0)$ whose distance to c in the curve graph is at most k and whose distance to gc is at most one. From this the lemma follows. $\square$

As a corollary, we obtain.

Corollary 8.5. *For every $\epsilon > 0$ there is a number $m(\epsilon) > 0$ with the following property. Let $H \subset \mathcal{M}(S)$ be the set of conjugacy classes of pseudo-Anosov elements whose translation length on the curve graph is at most $m(\epsilon)$. For $r > 0$ let $n_H(r)$ be the number of all elements in H which correspond to closed geodesics in moduli space of length at most r; then*

$$\limsup_{r \to \infty} \frac{1}{r} \log n(r) = \max\{6g - 6 + 2m, \limsup_{r \to \infty} \frac{1}{r} \log n_H(r)\}.$$

Proof. The corollary follows from Lemma 8.4 and Lemma 7.1. $\square$

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